FINITE ELEMENTS   HOMEWORK #1

Basics of FE

1. $-u'' = f$ in $(0, 1)$

arbitrary weight

$$\frac{\partial^2 u}{\partial x^2} = -f \Rightarrow \int_\Omega w_p \frac{\partial^2 u}{\partial x^2} = -\int_\Omega w_p \cdot f \qquad \boxed{\text{forma débil}}$$

$$\Rightarrow +\underbrace{\int_{\partial\Omega} w_p \frac{\partial u}{\partial x}}_{\text{boundary conditions}} - \int_\Omega \frac{\partial u}{\partial x} \cdot \frac{\partial w_p}{\partial x} = -\int_\Omega w_p \cdot f$$

for the Neumann case

Discretizing u as a sumation of shape functions, and imposing Galerkin ($w_p = N_p$), we obtain:

$$\underbrace{N_p \sum^n \frac{\partial N_i}{\partial x} u_i}_{\text{boundary conditions}} \Big|_0^1 - \int_\Omega \sum^n \frac{\partial N_i}{\partial x} u_i \frac{\partial N_p}{\partial x} = -\int_\Omega N_p \cdot f$$

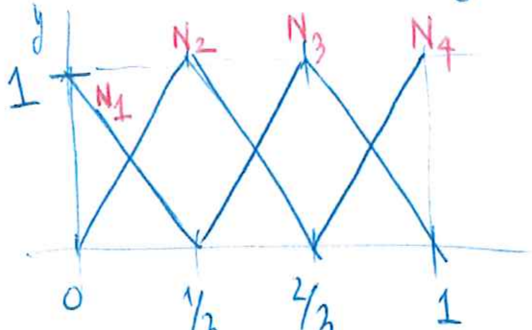
being u_i the FFEE descomposition of u.

2) Developing the previous expression, we have the following structure:

$$\begin{array}{c}\boxed{\text{Boundary conditions}}\\ \boxed{\text{Dirichlet}}\end{array} + \underbrace{\begin{pmatrix} \underset{= k_{ij}}{\int \frac{\partial N_i}{\partial x}\cdot\frac{\partial N_j}{\partial x}} \end{pmatrix}}_{\underline{\underline{K}}} \cdot \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}_{\hat{u}} = \underbrace{\begin{pmatrix} \int N_i f \end{pmatrix}}_{\hat{f}}$$

This is an standard system of equations with its appropriate boundary conditions.

3) We have the following shape functions:



$$N_1(x) = 1 - 3x$$

$$N_2(x) = \begin{cases} 3x & \text{in } (0, 1/3) \\ 2 - 3x & \text{in } (1/3, 2/3) \end{cases}$$

$$N_3(x) = \begin{cases} -1 + 3x & \text{in } (1/3, 2/3) \\ 3 - 3x & \text{in } (2/3, 1) \end{cases}$$

$$N_4(x) = -2 + 3x \quad \text{in } (2/3, 1)$$

So it is necessary to solve the following system of equations:

$$\begin{pmatrix} \int_0^1 N_1' N_1' & \int_0^1 N_2' N_1' & 0 & 0 \\ & \cdots & & 0 \\ & & \cdots & \\ (\text{sym}) & & & \int_0^1 N_4' N_4' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \int_0^1 N_1(x) \cdot \operatorname{sen}(x)\, dx \\ \vdots \\ \int_0^1 N_4(x) \cdot \operatorname{sen}(x)\, dx \end{pmatrix} \quad \text{and} \quad \boxed{\text{B.B:}} \begin{cases} \boxed{u_1 = 0} \\ \boxed{u_4 = 3} \end{cases}$$

Doing the necessary computations, we obtain:

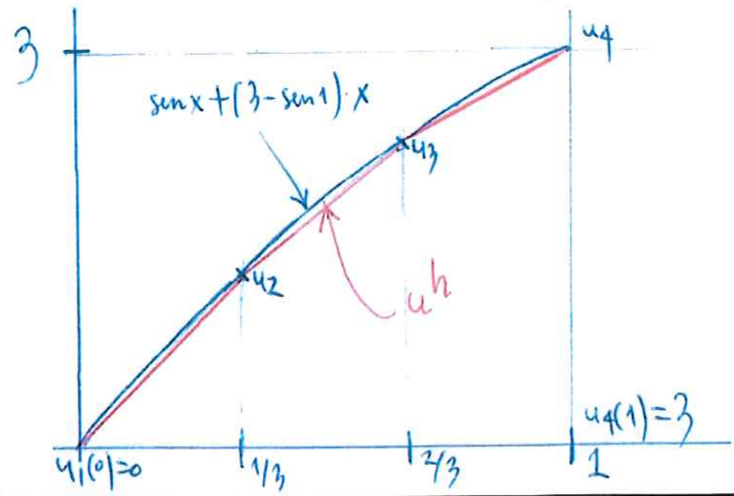
$$\begin{pmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.0184 \\ 0.1080 \\ 0.2042 \\ 0.1290 \end{pmatrix}$$

Eliminating both first and fourth files and columns and taking into account the boundary conditions information, we have:

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0.1080 + \underbrace{0}_{\boxed{u_1}} \\ 0.2042 + \underbrace{9}_{\boxed{3u_4}} \end{pmatrix}$$

$\boxed{\text{B.C.}}$

So, $\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1.0467 \\ 2.057 \end{pmatrix}$, and finally $\boxed{\bar{u}^h = \begin{pmatrix} 0 \\ 1.0467 \\ 2.057 \\ 3 \end{pmatrix}}$



Analytical and FEM approximation comparison