Finite Elements Method

Plane Elasticity

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Our aim is to study the deformation of a triangular thin plate under its self weight and an imposed vertical displacement δ on the tip. Figure 1 shows that, due to the symmetry of the problem, it is enough to analyze the left half of the domain (Ω).

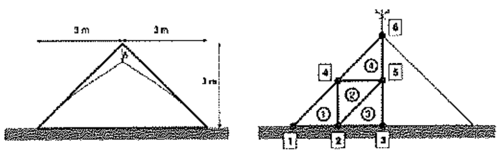

Figure 1: Geometry of the structure and triangular elements mesh, where node numbers are in the squared boxes and element numbers are in circles.

We will use 2D triangular elements to work out the problem under the assumption of plane stress, since the thickness t of the plate is much smaller than the other two dimensions. In fact, we set $t = 1$. Moreover, all the loads are contained in the middle plane of the structure, so it is worth to make this assumption.

1 Strong form equation and boundary conditions

Assuming that the problem is stationary (so there is no time dependence), the equilibrium between the body forces and the stress field related to σ is given by:

$$\nabla \cdot \sigma + \mathbf{b} = \mathbf{0} \text{ in } \Omega \tag{1}$$

In our case, $\mathbf{b} = (0, -\rho g)$, because the structure is under its self-weight, and $\sigma_z = 0$ because we assume plane stress. Thus, the only non-zero components of σ are σ_x, σ_y and $\tau_{xy} = \tau_{yx}$.

The relation between the stress field σ and the strain vector ε is given by the elastic matrix $\mathbf{D}$,

$$\sigma = \mathbf{D}\varepsilon \tag{2}$$

where

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{12} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \varepsilon = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right]^T = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^T \equiv \mathbf{L}\mathbf{u} \tag{3}$$

E is the Young's modulus, ν is the Poisson's ratio, and $u = u(x,y)$ and $v = v(x,y)$ are the components of the displacement vector $\mathbf{u}(x,y)$, which tell us the displacements of a point in directions x and y, respectively. Moreover,

$$\mathbf{L} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_y \\ \partial_y & \partial_x \end{bmatrix} \tag{4}$$

which allows us to rewritte (1) as

$$\mathbf{L}^T\sigma + \mathbf{b} = \mathbf{0} \text{ in } \Omega \tag{5}$$

using (2) and (3), it becomes

$$\mathbf{L}^T\mathbf{D}\mathbf{L}\mathbf{u} + \mathbf{b} = 0 \text{ in } \Omega \tag{6}$$

which is nothing but the strong form of the problem, because it involves second derivatives of $\mathbf{u}$.

Concerning the boundary conditions ($\Gamma_u \bigcup \Gamma_t = \partial\Omega,\ \Gamma_u \bigcap \Gamma_t = 0$), we have:

- Dirichlet boundary conditions (Γ_u):

$$\mathbf{u} = 0 \text{ on } \Gamma_u \tag{7}$$

  In particular, the prescribed nodal displacements will be:

$$\mathbf{u}_i\,(x,y) = (u_i, v_i) = \mathbf{0} \quad i = 1,2,3 \qquad u_5 = u_6 = 0 \qquad v_6 = -\delta \tag{8}$$

- Neumann boundary conditions (Γ_t). Considering the prescribed traction vector $\mathbf{t} = \mathbf{0}$,

$$\mathbf{M}^T\sigma = \mathbf{t} = \mathbf{0} \text{ on } \Gamma_t \rightarrow \begin{cases} \sigma_x n_x + \tau_{xy} n_y = t_x = 0 \\ \tau_{yx} n_x + \sigma_y n_y = t_y = 0 \end{cases} \tag{9}$$

$$\mathbf{M} = \begin{pmatrix} n_x & 0 \\ 0 & n_y \\ n_y & n_x \end{pmatrix} \tag{10}$$

  where n_x, n_y are the components of the normal vector $\hat{\mathbf{n}}$. The Dirichlet boundary conditions are applied at $\mathbf{u}_4 = (u_4, v_4)\,, v_5$. For this last one, we will only have $t_y = 0$, since u_5 is prescribed.

2 Nodal coordinates and conectivity matrix

The mesh shown in Figure 1 has 6 nodes and 4 triangular elements. The array of nodal coordinates $\mathbf{X}$ is:

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}^T = \begin{bmatrix} 0 & 1.5 & 3 & 1.5 & 3 & 3 \\ 0 & 0 & 0 & 1.5 & 1.5 & 3 \end{bmatrix}^T \tag{11}$$

where each column of (11) corresponds to one node of the mesh. The relation between the global nodal representation and the local one is given by the connectivity matrix $\mathbf{T}$,

$$\mathbf{T} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix} \tag{12}$$

where each row corresponds to one element and each column corresponds to one local node. For instance, for the first element, we can express global node 2 as the local node 1, global node 4 as the local node 2, and global node 1 as the local node 3. The local node 1 of each element corresponds to the node in the right angle vertex. This selection has been in order to simplify the computations.

3 Setting the system of equations

Our aim is to compute the unknown displacements in the global nodes 4 and 5 trough a system of equations after applying the finite element method:

$$\mathbf{K}\mathbf{a} = \mathbf{f} \tag{13}$$

where $\mathbf{K}$ is the Stiffness matrix, $\mathbf{a}$ contains the unknowns (displacements) and $\mathbf{f}$ is related to the forces acting on the system [1]. To do this, we have to rewrite the strong form of the equilibrium equation into its weak form. Therefore, we will get the equilibrium equation in terms of the first derivatives instead of in the second derivatives, and we will use C^0 continuous elements. This transformation can be done by applying the virtual work principle.

[1]See equations (22), (27) and (28) for further details.

Once the weak form is found, it is discretized using FEM:

$$\mathbf{u} \simeq \mathbf{u}^h = \left(u^h, v^h\right) = \left(\sum_i N_i u_i, \sum_i N_i v_i\right) \tag{14}$$

where N_i are the shape functions for a 3-noded triangular element.

For this kind of shape functions, it is easy to write an element stiffness submatrix $K_{ij}^{(e)}$:

$$K_{ij}^{(e)} = \iint_{A^{(e)}} \mathbf{B}_i{}^T \mathbf{D} \mathbf{B}_j t \mathrm{d}A = \iint_{A^{(e)}} \mathbf{B}_i{}^T \mathbf{D} \mathbf{B}_j \mathrm{d}A \tag{15}$$

where there are as many $\mathbf{B}_i$ as element nodes. In the case of a 3-noded triangle,

$$\mathbf{B}^{(e)} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & 0 & \vdots & b_2 & 0 & \vdots & b_3 & 0 \\ 0 & c_1 & \vdots & 0 & c_2 & \vdots & 0 & c_3 \\ c_1 & b_1 & \vdots & c_2 & b_2 & \vdots & c_3 & b_3 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{B}_1 & \vdots & \mathbf{B}_2 & \vdots & \mathbf{B}_3 \end{bmatrix} \tag{16}$$

with $b_i = y_j - y_k$, $c_i = x_k - x_j$, and $A^{(e)} = \frac{1.5 \cdot 1.5}{2} = 1.125\ \mathrm{m}^2$ the triangular element's area. Since we have four elements, this means that we will have four $\mathbf{B}^{(e)}$, e=1,2,3,4:

$$B^{(1)} = \frac{1}{4.5} \begin{bmatrix} 1.5 & 0 & \vdots & 0 & 0 & \vdots & -1.5 & 0 \\ 0 & -1.5 & \vdots & 0 & 1.5 & \vdots & 0 & 0 \\ -1.5 & 1.5 & \vdots & 1.5 & 0 & \vdots & 0 & -1.5 \end{bmatrix} \tag{17}$$

$$B^{(2)} = \frac{1}{4.5} \begin{bmatrix} -1.5 & 0 & \vdots & 0 & 0 & \vdots & 1.5 & 0 \\ 0 & 1.5 & \vdots & 0 & -1.5 & \vdots & 0 & 0 \\ 1.5 & -1.5 & \vdots & -1.5 & 0 & \vdots & 0 & 1.5 \end{bmatrix} \tag{18}$$

$$B^{(3)} = \frac{1}{4.5} \begin{bmatrix} 1.5 & 0 & \vdots & 0 & 0 & \vdots & -1.5 & 0 \\ 0 & -1.5 & \vdots & 0 & 1.5 & \vdots & 0 & 0 \\ -1.5 & 1.5 & \vdots & 1.5 & 0 & \vdots & 0 & -1.5 \end{bmatrix} \tag{19}$$

$$B^{(4)} = \frac{1}{4.5} \begin{bmatrix} 1.5 & 0 & \vdots & 0 & 0 & \vdots & -1.5 & 0 \\ 0 & -1.5 & \vdots & 0 & 1.5 & \vdots & 0 & 0 \\ -1.5 & 1.5 & \vdots & 1.5 & 0 & \vdots & 0 & -1.5 \end{bmatrix} \tag{20}$$

And then, using (15), one can find $K_{11}^{(1)}$, $K_{21}^{(3)}$, $K_{32}^{(4)}$, etc.

Using the connectivity matrix one can make the assembly process in order to find the Stiffness matrix $\mathbf{K}$,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} & \mathbf{K}_{15} & \mathbf{K}_{16} \\ & \mathbf{K}_{22} & \mathbf{K}_{23} & \mathbf{K}_{24} & \mathbf{K}_{25} & \mathbf{K}_{26} \\ & & \mathbf{K}_{33} & \mathbf{K}_{34} & \mathbf{K}_{35} & \mathbf{K}_{36} \\ & \text{Sym.} & & \mathbf{K}_{44} & \mathbf{K}_{45} & \mathbf{K}_{46} \\ & & & & \mathbf{K}_{55} & \mathbf{K}_{56} \\ & & & & & \mathbf{K}_{66} \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} \mathbf{K}_{33}^{(1)} & \mathbf{K}_{31}^{(1)} & \mathbf{0} & \mathbf{K}_{32}^{(1)} & \mathbf{0} & \mathbf{0} \\ & \mathbf{K}_{11}^{(1)} + \mathbf{K}_{22}^{(2)} + \mathbf{K}_{33}^{(3)} & \mathbf{K}_{31}^{(3)} & \mathbf{K}_{12}^{(1)} + \mathbf{K}_{21}^{(2)} & \mathbf{K}_{23}^{(2)} + \mathbf{K}_{32}^{(3)} & \mathbf{0} \\ & & \mathbf{K}_{11}^{(3)} & \mathbf{0} & \mathbf{K}_{12}^{(3)} & \mathbf{0} \\ & \text{Sym.} & & \mathbf{K}_{22}^{(1)} + \mathbf{K}_{11}^{(2)} + \mathbf{K}_{33}^{(4)} & \mathbf{K}_{13}^{(2)} + \mathbf{K}_{31}^{(4)} & \mathbf{K}_{32}^{(4)} \\ & & & & \mathbf{K}_{33}^{(2)} + \mathbf{K}_{22}^{(3)} + \mathbf{K}_{11}^{(4)} & \mathbf{K}_{12}^{(4)} \\ & & & & & \mathbf{K}_{22}^{(4)} \end{bmatrix} \tag{22}$$

Once $\mathbf{K}$ has been found, we proceed to compute $\mathbf{f}$, the force vector. As it happened before, we will compute f in terms of $\mathbf{f}^{(e)}$, which is

$$\mathbf{f}^{(e)} = \mathbf{f}_\varepsilon^{(e)} + \mathbf{f}_\sigma^{(e)} + \mathbf{f}_b^{(e)} + \mathbf{f}_t^{(e)} \quad e = 1,2,3,4 \tag{23}$$

where $\mathbf{f}_\varepsilon^{(e)}, \quad \mathbf{f}_\sigma^{(e)}, \quad \mathbf{f}_b^{(e)}, \quad \mathbf{f}_t^{(e)}$ are the equivalent nodal force vectors due to initial strains, initial stresses, body forces and surface tensions, respectively. In our case, we will only consider body forces, so (23) becomes:

$$\mathbf{f}^{(e)} = \mathbf{f}_b^{(e)} = \iint_{A^{(e)}} \mathbf{N}^T \mathbf{b} t dA \quad e = 1,2,3,4 \tag{24}$$

where, if we consider that the body forces are uniformly distributed over the element, each node has a contribution

$$\mathbf{f}_{b_i}^{(e)} = \frac{(At)^{(e)}}{3} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{A^{(e)}}{3} \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} = 0.375 \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} \tag{25}$$

because we have particularized for $t = 1$ and for the self-weight with gravity acting along the y-axis as the body force. Now, in a similar way as for $\mathbf{K}$, we set $\mathbf{f}$:

$$\mathbf{f} = \left[\mathbf{f}_1, \quad \mathbf{f}_2, \quad \mathbf{f}_3, \quad \mathbf{f}_4, \quad \mathbf{f}_5, \quad \mathbf{f}_6\right]^T \tag{26}$$

$$= \left[\mathbf{f}_3^{(1)}, \quad \mathbf{f}_1^{(1)} + \mathbf{f}_2^{(2)} + \mathbf{f}_3^{(3)}, \quad \mathbf{f}_1^{(3)}, \quad \mathbf{f}_2^{(1)} + \mathbf{f}_1^{(2)} + \mathbf{f}_3^{(4)}, \quad \mathbf{f}_3^{(2)} + \mathbf{f}_2^{(3)} + \mathbf{f}_1^{(4)}, \quad \mathbf{f}_2^{(4)}\right]^T \tag{27}$$

Finally, concerning the node displacements,

$$\mathbf{a} = \left[\mathbf{u}_1, \quad \mathbf{u}_2, \quad \mathbf{u}_3, \quad \mathbf{u}_4, \quad \mathbf{u}_5, \quad \mathbf{u}_6\right]^T = \left[0, \quad 0, \quad 0, \quad \{u_4 \quad v_4\}, \quad \{0 \quad v_5\}, \quad \{0 \quad -\delta\}\right]^T \tag{28}$$

Replacing (22), (27) and (28) into (13),

$$\begin{bmatrix} \mathbf{K}_{33}^{(1)} & \mathbf{K}_{31}^{(1)} & \mathbf{0} & \mathbf{K}_{32}^{(1)} & \mathbf{0} & \mathbf{0} \\ & \mathbf{K}_{11}^{(1)} + \mathbf{K}_{22}^{(2)} + \mathbf{K}_{33}^{(3)} & \mathbf{K}_{31}^{(3)} & \mathbf{K}_{12}^{(1)} + \mathbf{K}_{21}^{(2)} & \mathbf{K}_{23}^{(2)} + \mathbf{K}_{32}^{(3)} & \mathbf{0} \\ & & \mathbf{K}_{11}^{(3)} & \mathbf{0} & \mathbf{K}_{12}^{(3)} & \mathbf{0} \\ & \text{Sym.} & & \mathbf{K}_{22}^{(1)} + \mathbf{K}_{11}^{(2)} + \mathbf{K}_{33}^{(4)} & \mathbf{K}_{13}^{(2)} + \mathbf{K}_{31}^{(4)} & \mathbf{K}_{32}^{(4)} \\ & & & & \mathbf{K}_{33}^{(2)} + \mathbf{K}_{22}^{(3)} + \mathbf{K}_{11}^{(4)} & \mathbf{K}_{12}^{(4)} \\ & & & & & \mathbf{K}_{22}^{(4)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -\delta \end{bmatrix} = 0.375 \begin{bmatrix} 0 \\ -\rho g \\ 0 \\ -3\rho g \\ 0 \\ -\rho g \\ 0 \\ -3\rho g \\ 0 \\ -3\rho g \\ 0 \\ -\rho g \end{bmatrix} \tag{29}$$

which is a 12×12 system of equations. Nevertheless, since the displacements for some rows are already known, it turns to be the following 6×6 system:

$$\begin{bmatrix} \mathbf{K}_{22}^{(1)} + \mathbf{K}_{11}^{(2)} + \mathbf{K}_{33}^{(4)} & \mathbf{K}_{13}^{(2)} + \mathbf{K}_{31}^{(4)} & \mathbf{K}_{32}^{(4)} \\ & \mathbf{K}_{33}^{(2)} + \mathbf{K}_{22}^{(3)} + \mathbf{K}_{11}^{(4)} & \mathbf{K}_{12}^{(4)} \\ \text{Sym.} & & \mathbf{K}_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -\delta \end{bmatrix} = 0.375 \begin{bmatrix} 0 \\ -3\rho g \\ 0 \\ -3\rho g \\ 0 \\ -\rho g \end{bmatrix} \tag{30}$$

Notice that this system will be, in fact, a 3×3 one, because the node displacements u_5, u_6 and v_6 are also prescribed. Therefore, the system of equations will have three degrees of freedom.

We will set the 3×3 system in the following section, where we will compute explicitly the values of $\mathbf{K}_{ij}^{(e)}$.

4 Solving the system of equations

We are now asked to solve (30) with $E = 10$ GPa $= 10^{10}$ Pa, $\nu = 0.2$, $\delta = 10^{-2}$ m and $\rho g = 10^3$ N/m^2.

First of all, let's compute all the necessary $K_{ij}^{(e)}$. To do so, we can use (15)

$$K_{ij}^{(e)} = \int\int_{A^{(e)}} \mathbf{B}_i^{(e)^T} \mathbf{D} \mathbf{B}_j^{(e)} t \mathrm{d}A = \int\int_{A^{(e)}} \mathbf{B}_i^{(e)^T} \mathbf{D} \mathbf{B}_j^{(e)} \mathrm{d}A \tag{31}$$

However, since its integrand is constant, (31) becomes:

$$K_{ij}^{(e)} = \left(\frac{1}{4A}\right)^{(e)} \begin{bmatrix} b_ib_jd_{11} + c_ic_jd_{33} & b_ic_jd_{12} + b_jc_id_{33} \\ c_ib_jd_{21} + b_ic_jd_{33} & b_ib_jd_{33} + c_ic_jd_{22} \end{bmatrix} = \frac{1}{4.5} \begin{bmatrix} b_ib_jd_{11} + c_ic_jd_{33} & b_ic_jd_{12} + b_jc_id_{33} \\ c_ib_jd_{21} + b_ic_jd_{33} & b_ib_jd_{33} + c_ic_jd_{22} \end{bmatrix} \tag{32}$$

We will now compute the $K_{ij}^{(e)}$ of (30) using (32) and taking into account (17), (18), (19), (20) and (3). After that, replacing the values into (30), we get

$$\frac{10^{10}}{1.92} \begin{bmatrix} 2.8 & -0.6 & -2 & 0.6 & 0 & -0.2 \\ -0.6 & 2.8 & 0.6 & -0.8 & -0.4 & 0 \\ -2 & 0.6 & 2.8 & -0.6 & -0.4 & 0.2 \\ 0.6 & -0.8 & -0.6 & 2.8 & 0.4 & -1 \\ 0 & -0.4 & -0.4 & 0.4 & 0.4 & 0 \\ -0.2 & 0 & 0.2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -1125 \end{bmatrix} \tag{33}$$

and now, thanks to the prescribed displacements in some nodes, it is straightforward to rewrite as

$$\frac{10^{10}}{1.92} \begin{bmatrix} 2.8 & -0.6 & 0.6 \\ -0.6 & 2.8 & -0.8 \\ 0.6 & -0.8 & 2.8 \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -\frac{0.2 \cdot 10^8}{1.92} \\ -1125 \\ -1125 - \frac{10^8}{1.92} \end{bmatrix} \tag{34}$$

After solving this system we find the unknown nodal displacements of the plate:

$$u_4 = -0.000128205128205 \text{ m} \simeq -0.128 \text{ mm} \tag{35}$$

$$v_4 = -0.001132586632479 \text{ m} \simeq -1.133 \text{ mm} \tag{36}$$

$$v_5 = -0.003867629367521 \text{ m} \simeq -3.868 \text{ mm} \tag{37}$$

And finally, from (14), we compute $\mathbf{u}^h$.

$$\mathbf{u}^h = \left(\sum_i N_i u_i, \sum_i N_i v_i\right) = \left(\mathbf{u}_4 N_2^{(1)} + \mathbf{u}_4 N_1^{(2)} + \mathbf{u}_4 N_3^{(4)} + \mathbf{u}_5 N_3^{(2)} + \mathbf{u}_5 N_1^{(4)} + \mathbf{u}_5 N_2^{(3)} + \mathbf{u}_6 N_2^{(4)}\right)^T \tag{38}$$

with $\mathbf{u}_4 = (-0.000128205128205, -0.001132586632479)$, $\mathbf{u}_5 = (0, -0.003867629367521)$, $\mathbf{u}_6 = (0, -10^{-2})$. The other nodal displacements are 0. However, we first have to compute the shape functions, with:

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y) = \frac{1}{2.25} (a_i + b_i x + c_i y) \tag{39}$$

where:

$$a_i = x_j y_k - x_k y_j \quad b_i = y_j - y_k \quad c_i = x_k - x_j \tag{40}$$

which depend on the element. Finally, replacing this in (38),

$$\mathbf{u}^h = \left(\sum_i N_i u_i, \sum_i N_i v_i\right) = \left(\mathbf{u}_4 N_2^{(1)} + \mathbf{u}_4 N_1^{(2)} + \mathbf{u}_4 N_3^{(4)} + \mathbf{u}_5 N_3^{(2)} + \mathbf{u}_5 N_1^{(4)} + \mathbf{u}_5 N_2^{(3)} + \mathbf{u}_6 N_2^{(4)}\right)^T \tag{41}$$

$$= \left(\frac{1}{2.25} [1.5y\mathbf{u}_4 + (2.25 - 1.5x + 1.5y)\, \mathbf{u}_4 + (4.5 - 1.5x)\, \mathbf{u}_4 + (-2.25 + 1.5x)\, \mathbf{u}_5 \right. \tag{42}$$

$$\left. + (1.5x - 1.5y)\, \mathbf{u}_5 + 1.5y\mathbf{u}_5 + (-2.25 + 1.5y)\, \mathbf{u}_6]\right)^T$$

Notice that each of this shape functions is 1 on its node (within its element) and 0 at the other nodes. For instance, $N_3^{(2)} = \frac{1}{2.25} (-2.25 + 1.5x)\Big|_{3^{(2)}} = \frac{1}{2.25} (-2.25 + 1.5 \cdot 3) = 1$, but $N_3^{(2)} = \frac{1}{2.25} (-2.25 + 1.5x)\Big|_{1^{(2)}} =$

$\frac{1}{2.25}(-2.25+1.5\cdot 1.5)=0$, $N_3^{(2)}=\frac{1}{2.25}(-2.25+1.5x)\Big|_{2^{(2)}}=\frac{1}{2.25}(-2.25+1.5\cdot 1.5)=0$. Now, replacing the values of $\mathbf{u}_4$, $\mathbf{u}_5$, $\mathbf{u}_6$,

$$\mathbf{u}^h=\Big[5.698\cdot 10^{-5}(6.75-3x+3y)\,,(6.75-3x+3y)\,5.0337\cdot 10^{-4} \tag{43}$$

$$-1.7189\cdot 10^{-3}(-2.25+3x)+4.4445\cdot 10^{-2}(-2.25+1.5y)\Big]^T$$

Finally, we just have to concern about the elements that share an edge, and the solution there must come from one of the elements, not both. This is why we normalize (43) with a factor $N_e=1,2,3$, since up to three elements can share a certain point (x,y). For instance, if $N_e=2$, the factor $1/2$ takes into account only the contribution of a single element instead of 2. The value that N_e takes a decision of the reader, depending on where he/she wants to compute the solution.

To sum up,

$$\mathbf{u}^h=\frac{1}{N_e}\Big[5.698\cdot 10^{-5}(6.75-3x+3y)\,,(6.75-3x+3y)\,5.0337\cdot 10^{-4}+-1.7189\cdot 10^{-3}(-2.25+3x) \tag{44}$$

$$+4.4445\cdot 10^{-2}(-2.25+1.5y)\Big]^T \qquad N_e=1,2,3$$