10,5

FINITE ELEMENTS January 20, 2014

Time allowed: 2:00 hours

All the exam should be developed on this sheets. No additional sheets will be corrected.

The Poisson equation is solved using the Finite Element Method in a rectangular domain of height H and width $3H$. The problem is stated as follows

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= x \quad \text{on } \Gamma_D \subset \partial\Omega \end{aligned}$$

where the source term f is constant and the Dirichlet boundary Γ_D includes the two lateral sides of the domain ($x = 0$ and $x = 3H$). In the rest of the boundary natural boundary conditions (homogeneous Neumann) are imposed.

The mesh is constituted by 8 nodes and 6 triangular three-noded elements and it is characterized by the following nodal coordinates and connectivity:

$$\mathtt{X} = \begin{bmatrix} H & 0 \\ 2H & 0 \\ 2H & -H \\ H & -H \\ 0 & -H \\ 0 & 0 \\ 3H & 0 \\ 3H & -H \end{bmatrix} \quad \text{and } \mathtt{T} = \begin{bmatrix} 6 & 1 & 5 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 2 \\ 2 & 7 & 3 \\ 7 & 8 & 3 \end{bmatrix}$$

a) Represent graphically the mesh, numbering the elements and the nodes (both local and global node numbering).

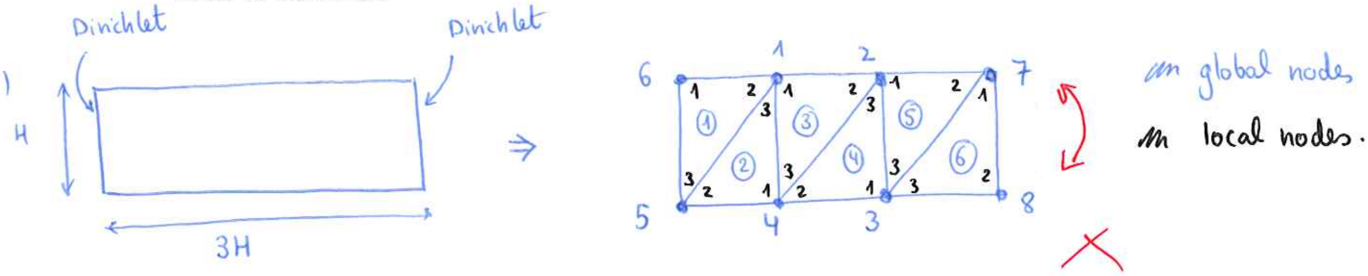
b) Compute the 3×3 elementary stiffness matrix for the first element (with vertices (0,0), (H,0) and (0,H)). Give the expressions of the shape functions and their derivatives.

c) Noting that all the elements in the mesh are similar to the first element, assemble the global 8×8 stiffness matrix without accounting for Dirichlet boundary conditions.

d) Repeat questions **b** and **c** for the force term vector.

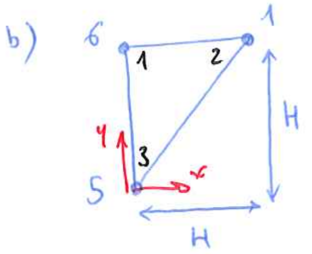
e) Use the Dirichlet boundary conditions and find the reduced 4×4 linear system of equations to be solved.

a)



$$a_1 = x_2 \cdot y_3 - x_3 \cdot y_2 = 0$$
$$a_2 = x_3 \cdot y_1 - x_1 \cdot y_3 = 0$$
$$a_3 = x_1 \cdot y_2 - x_2 \cdot y_1 = -H^2$$

b)



$$A^{(e)} = \frac{H \cdot H}{2} = \frac{H^2}{2}.$$

$$b_1 = y_2 - y_3 = +H \qquad c_1 = x_3 - x_2 = -H$$
$$b_2 = y_3 - y_1 = -H \qquad c_2 = x_1 - x_3 = 0$$
$$b_3 = y_1 - y_2 = 0 \qquad c_3 = x_2 - x_1 = H$$

$$K^{(1)} = \frac{1}{4 \cdot H^2/2} \begin{bmatrix} 2H^2 & -H^2 & -H^2 \\ -H^2 & H^2 & 0 \\ -H^2 & 0 & H^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Shape functions: $N_i^{(e)}(x,y) = \frac{1}{2A^{(e)}} \left[a_i^{(e)} + b_i^{(e)} x + c_i^{(e)} y \right]$

$$N_1^{(1)}(x,y) = \frac{1}{H^2} \left[+Hx - Hy \right] = \frac{-1}{H}(x+y) \qquad \frac{\partial N_1^{(1)}}{\partial x} = \frac{+1}{H} \qquad \frac{\partial N_1^{(1)}}{\partial y} = \frac{-1}{H}$$

$$N_2^{(1)}(x,y) = \frac{1}{H^2} \left[-Hx \right] = -\frac{x}{H} \qquad \frac{\partial N_2^{(1)}}{\partial x} = -\frac{1}{H} \qquad \frac{\partial N_2^{(1)}}{\partial y} = 0$$

$$N_3^{(1)}(x,y) = \frac{1}{H^2} \left[-H^2 + Hy \right] = \frac{1}{H} \left[-1 + y \right] \qquad \frac{\partial N_3^{(1)}}{\partial x} = 0 \qquad \frac{\partial N_3^{(1)}}{\partial y} = \frac{1}{H}$$

c) Thanks to the connectivity: $T = \begin{bmatrix} 6\,1\,5 \\ 4\,5\,1 \\ 1\,2\,4 \\ 3\,4\,2 \\ 2\,7\,3 \\ 7\,8\,3 \end{bmatrix} \begin{matrix} \text{1st el} \\ \text{2nd el.} \\ \text{3rd el} \\ \text{4th el} \\ \text{5th el.} \\ \text{6th el.} \end{matrix}$

$$\begin{bmatrix} K_{22}^{(1)} + K_{33}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & 0 & K_{31}^{(2)} + K_{13}^{(3)} & K_{32}^{(1)} + K_{32}^{(2)} & K_{21}^{(1)} & 0 & 0 \\ & K_{22}^{(3)} + K_{33}^{(4)} + K_{11}^{(5)} & K_{31}^{(4)} + K_{13}^{(5)} & K_{23}^{(3)} + K_{32}^{(4)} & 0 & 0 & K_{12}^{(5)} & 0 \\ & & K_{11}^{(4)} + K_{33}^{(5)} + K_{33}^{(6)} & K_{12}^{(4)} & 0 & 0 & K_{32}^{(5)} & K_{32}^{(6)} \\ & & & K_{11}^{(2)} + K_{33}^{(3)} + K_{22}^{(4)} & K_{12}^{(2)} & 0 & 0 & 0 \\ & \text{symm.} & & & K_{33}^{(1)} + K_{22}^{(2)} & K_{31}^{(1)} & 0 & 0 \\ & & & & & K_{11}^{(1)} & 0 & 0 \\ & & & & & & K_{22}^{(5)} + K_{11}^{(6)} & K_{12}^{(6)} \\ & & & & & & & K_{22}^{(6)} \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 4 & -1 & 0 & -2 & 0 & -1 & 0 & 0 \\ & 4 & -2 & 0 & 0 & 0 & -1 & 0 \\ & & 4 & -1 & 0 & 0 & 0 & 0 \\ & & & 4 & -1 & 0 & 0 & 0 \\ & \text{symm.} & & & 2 & -1 & 0 & 0 \\ & & & & & 2 & 0 & 0 \\ & & & & & & 3 & -1 \\ & & & & & & & -1 \end{bmatrix}$$

d) For the force term vector:

$$f_Q^{(e)} = \frac{f \cdot A^{(e)}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{f\cdot H^2}{6}\begin{bmatrix}1\\1\\1\end{bmatrix}$$, and we have ~~nota~~ natural

due to f in the equation

boundary conditions $\Rightarrow$ $f^{(e)} = f_Q^{(e)} = \frac{fH^2}{6}\begin{bmatrix}1\\1\\1\end{bmatrix}$.

And assembling the force vector:

$$[f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7 \quad f_8]^T =$$

$$= [f_2^{(1)} + f_3^{(2)} + f_1^{(3)},\ f_2^{(3)} + f_3^{(4)} + f_1^{(5)},\ f_1^{(4)} + f_3^{(5)} + f_3^{(6)},\ f_1^{(2)} + f_3^{(3)} + f_2^{(4)},$$

$$f_3^{(1)} + f_2^{(2)},\ f_1^{(1)},\ f_1^{(6)} + f_2^{(5)},\ f_2^{(6)}]^T =$$

$$= \frac{fH^2}{6}[3, 3, 3, 3, 2, 1, 2, 1]$$

e)

$$\frac{1}{2}\begin{bmatrix} 4 & -1 & 0 & -2 & 0 & -1 & 0 & 0 \\ & 4 & -2 & 0 & 0 & 0 & -1 & 0 \\ & & 4 & -1 & 0 & 0 & 0 & 0 \\ & & & 4 & -1 & 0 & 0 & 0 \\ & \text{symm.} & & & 2 & -1 & 0 & 0 \\ & & & & & 2 & 0 & 0 \\ & & & & & & 3 & -1 \\ & & & & & & & -1 \end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\\U_4\\0\\0\\3H\\3H\end{bmatrix} = \begin{bmatrix}3\\3\\3\\3\\2+R_5\\1+R_6\\2+R_7\\1+R_8\end{bmatrix}\cdot\frac{fH^2}{6}$$

$R_5, \dots, R_8$ are the reactions on these nodes: 5,6,7,8.

$$\frac{1}{2}\begin{bmatrix} 4 & -1 & 0 & -2 \\ 1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ -2 & 0 & -1 & 4 \end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix} = \begin{bmatrix}3\\3-3H\\3\\3\end{bmatrix}$$

$\longrightarrow$ 4×4 system of equations to be solved. A posteriori we would find the reactions $R_5, \dots, R_8$.

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2. The following ODE has to be solved using the Finite Element Method

$$\begin{aligned} -u'' + u &= f \quad \text{in }]0,1[\\ u(0) &= 0 \quad \text{at } x = 0 \\ u'(1) &= \alpha \quad \text{at } x = 1 \end{aligned}$$

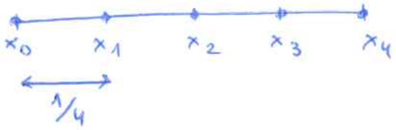
with a uniform discretization $\{x_0, x_1, x_3, x_4, x_4\}$, with $x_i = i/4$, for $i = 0, 1, \ldots, 4$, and both a mesh of 4 linear elements and a mesh of 2 quadratic elements.

a) Find the weak form of the problem.

b) Obtain the general expression of the elementary matrices for linear and quadratic elements.

c) Assemble the global matrices in the two cases.

a) Weak form:



$$\int_0^1 w(-u''+u-f)\,dx + \bar{w}(u'-\alpha)\Big|_1 = 0$$

We integrate by parts the term with the second derivative:

$$\int_0^1 -wu'' = -w\cdot u'\Big|_0^1 + \int_0^1 w'u'\,dx \quad , \text{ so:}$$

$$-wu'\Big|_0^1 + \int_0^1 w'u'\,dx + \int_0^1 w(u-f)\,dx + \bar{w}u'\Big|_1 - \bar{w}\alpha\Big|_1 = 0 . \quad \text{If } w = +\bar{w},$$

$$\cancel{-wu'\Big|_1} + wu'\Big|_0 + \int_0^1 w'u'\,dx + \int_0^1 w(u-f)\,dx \cancel{+\bar{w}u'\Big|_1} - w\alpha\Big|_1 = 0 .$$

$$\int_0^1 w'u'\,dx + \int_0^1 wu = w\alpha\Big|_1 - \cancel{wu'\Big|_0} + \int_0^1 wf\,dx .$$

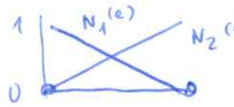
$u(0) = 0$ $\quad \forall w \mid w = 0$ on Γ_u,

$u'(1) - \alpha = 0$ $\quad u \in H_1(\Omega) \mid u = 0$ in Γ_d.

} Weak form of the problem.

b) $$K_{ij}^{(e)} = \int_0^1 \left(\frac{dN_i}{dx}\cdot\frac{dN_j}{dx} + N_i N_j \right) dx$$ → Expression for the element ij of the elementary matrix., with the Galerkin method $W_i = N_i$.

Linear elements



$N_1^{(e)} = -4x + 1$. $\qquad N_2^{(e)} = 4x$.

$$K_{11}^{(e)} = \int_0^1 \left(-4x\cdot(-4x) + (-4x+1)^2\right)dx = \int_0^1 (16x^2 + 16x^2 - 8x + 1)\,dx = \frac{32}{3}x^3 - 8\frac{x^2}{2} + x\Big|_0^1 = \frac{23}{3}$$

$$K_{21}^{(e)} = K_{12}^{(e)} = \int_0^1 \left(-4x\cdot 4 + 4x(-4x+1)\right)dx = \int_0^1 (-16x - 16x^2 + 4x)\,dx = \frac{-12x^2}{2} - 16\frac{x^3}{3}\Big|_0^1 = \frac{-34}{3}$$

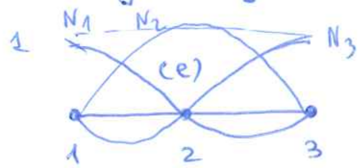
$$K_{22}^{(e)} = \int_0^1 (4\cdot 4 + 4x\cdot 4x)\,dx = \int_0^1 16 + 16x^2\,dx = 16x + \frac{16x^3}{3}\Big|_0^1 = \frac{64}{3}$$

So the elementary matrix is:

$$K^{(e)} = \begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 23 & -34 \\ -34 & 64 \end{bmatrix}$$

And all elements are equivalent, so $K_{ij}^{(1)} = K_{ij}^{(2)} = K_{ij}^{(3)} = K_{ij}^{(4)} \ \forall i,j$

<u>Quadratic elements</u>



← local numbering for one element (quadratic)

x: 0 1/4 1/2

$N_i = a_i x^2 + b_i x + c_i$

~~Working with an isoparametric formulation,~~

- $N_1 \Rightarrow a\cdot 0^2 + b\cdot 0 + c = 1 \rightarrow c = 1$

$a\cdot\frac{1}{16} + b\cdot\frac{1}{4} + c = 0$
$a\cdot\frac{1}{4} + b\cdot\frac{1}{2} + c = 0$

$\begin{cases} -a/8 - b/2 - 2c = 0 \\ a/4 + b/2 + c = 0 \end{cases}$

$a/8 - c = 0 \rightarrow a = 8,\ b = -6$

$N_1 = 8x^2 - 6x + 1$ //

- $N_2 \Rightarrow a\cdot 0^2 + b\cdot 0 + c = 0 \rightarrow c = 0$

$a\cdot 1/16 + b\cdot 1/4 + c = 1$
$a\cdot 1/4 + b\cdot 1/2 + c = 0$

$\begin{cases} -a/8 - b/2 - 2c = -2 \\ a/4 + b/2 + c = 0 \end{cases}$

$a/8 = -2 \rightarrow a = -16,\ b = 8$

$N_2 = -16x^2 + 8x.$

- $N_3 \Rightarrow a\cdot 0^2 + b\cdot 0 + c = 0 \rightarrow c = 0$

$a\cdot 1/16 + b/4 + c = 0$
$a/4 + b/2 + c = 1$

$\begin{cases} -a/8 - b/2 - 2c = 0 \\ a/4 + b/2 + c = 1 \end{cases}$

$a/8 = 1 \rightarrow a = 8,\ b = -2$

$N_3 = 8x^2 - 2x.$

$K_{11} = \int_0^1 (16x-6)^2 + (8x^2-6x+1)^2\,dx = 29{,}67$

$K_{12} = \int_0^1 (16x-6)(-32x+8) + (8x^2-6x+1)(-16x^2+8x)\,dx = -61{,}6 = K_{21}$

$K_{13} = \int_0^1 (16x-6)(16x-2) + (8x^2-6x+1)(8x^2-2x)\,dx = 35{,}8 = K_{31}$

$K_{22} = \int_0^1 (-16x^2+8x)^2 + (-32x+8)^2\,dx = 157{,}87$

$K_{23} = \int_0^1 (-16x^2+8x)(8x^2-2x) + (-32x+8)(16x-2)\,dx = -97{,}6$

$K_{33} = \int_0^1 (8x^2-2x)^2 + (16x-2)^2\,dx = 63{,}467$

All elements are equivalent, so $K_{ij}^{(1)} = K_{ij}^{(2)}$ $\forall i,j$

and this matrix should be filled with these values calculated above

c) Assembly:

$$K_{linear} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 \\ & K_{22}^{(1)}+K_{11}^{(3)} & K_{12}^{(2)} & 0 & 0 \\ & & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & 0 \\ \text{symm.} & & & K_{22}^{(3)}+K_{11}^{(4)} & K_{12}^{(4)} \\ & & & & K_{22}^{(4)} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 23 & -34 & 0 & 0 & 0 \\ & 87 & -34 & 0 & 0 \\ & & 87 & -34 & 0 \\ \text{symm.} & & & 87 & -34 \\ & & & & 64 \end{bmatrix};$$

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$$

$$K_{quadratic} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ & & K_{33}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ \text{sym.} & & & K_{22}^{(2)} & K_{23}^{(2)} \\ & & & & K_{33}^{(2)} \end{bmatrix}$$

3. COMMENT ON THE APPROPRIATENESS OF EACH OF THE FOLLOWING STATEMENTS, DISCUSSING WHICH PARTS ARE TRUE AND WHICH ARE FALSE.

a) The finite element approximation in the linear elastic problem in 1D should satisfy certain conditions which guarantee that as the mesh is refined, the numerical solution converges to the exact one. These are:

True 1. The continuity condition: the displacement should have C0 continuity within each element and along the element interfaces.

Elements must be compatible, so the displacement field for C_0 elements, or its first derivative for C^1 elements, must be continuous over interelemental boundaries. Moreover, the displacement should have C^0 continuity within each element.

True 2. The derivability condition: the derivatives of the function approximating the displacement should exist up to the order of the derivatives appearing in the element integrals.

For example, in the weak form of Poisson equation, only first derivatives appear. This means that we can use polynomials of at least first order, for which $\exists$ the first derivative.

False 3. The integrability condition: the integrals appearing in the element expressions must have a primitive function. If *m-th* order derivative of the displacement field appear in the weak form, the shape functions must be C^{m+1}

The integrals ~~have~~ must have a primitive function. However, if a m-th order derivative appears on the weak form, the shape functions must be C^{m-1} continuous. Ex: If a first derivative appears, C^0 linear splines are enough, we don't need C^2 functions.

True 4. The rigid body condition: when a rigid body motion is imposed, no strain should occur in the element. That is satisfied for a single element is the sum of the shape functions derivatives at any point is equal to 1

We choose the displacement field such that an element cannot strain if the nodal displacements are caused by a rigid body displacement, and if $\sum N_i = 1$ for a single element, this is satisfied.

False 5. The constant strain condition: the displacement function has to be such that is nodal displacements are compatible with a constant strain field, such constant strain should be obtained. This condition is incorporated by the rigid body condition.

The displacement function is choosen such that its nodal displacements are compatible with a constant strain field. However, this condition IS NOT incorporated by the rigid body condition, in fact it is the constant strain condition the one that incorporates the rigid body condition.

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b) About error estimates.

True 1. A priori error estimates are not well suited to compute the level of error of the FE approximation.

A priori error estimates only give information on how the error converges, but ~~...~~ they are not able to compute the value of the error itself. ⇒ they do not provide the exact error.

→ if one refines the mesh, for example.

✓

2. On the contrary, a posteriori error estimates deliver an upper-bound of the actual error, provided it is measured with the energy norm.

~~False~~ False

residual type

There are two types of a posteriori error estimates: implicit and explicit. For an implicit residual estimator, one can not bound the error, because this method is one-sided. ✓

However, an explicit residual method provides an upper-bound of it. no!

?

True 3. If the user prefers other measures of the error (different than the energy norm), a posteriori error estimates cannot be used anymore. no!

A posteriori - errors are based on energy measures $\|e^2\| = a(e,e) = R(e)$, being R the residual. Depending on the type of a posteriori error estimate, we approximate the residual by one way or another. Thus, if we don't use energy norm to measure the error, we cannot have a posteriori error estimate.

c) In the context of Structural Dynamics.

True 1. The modal approach is always preferred to the direct time integration because the number of d.o.f. is drastically reduced.

The modal approach says that ~~a~~ $M\ddot{a} + Ka = 0$ with N degrees of freedom is equivalent to a system of N equations. ?

False 2. The stability of the Newmark method is guaranteed independently of the selected Δt.

Newmark method has a limitation on the maximum time step Δt that can be choosen, it is not an unconditionally stable.

The stability properties of the Newmark method depen on the parameters γ, β, ~~θ~~, so the method is conditionally stable. ✓

False 3. In modal analysis, the convergence ratio is different for every eigen-mode, being better for those associated with lower eigen-frequencies

An implicit method with $\theta \geq 1/2$ is unconditionally stable, while an scheme with $\theta = 1/4$, ~~...~~ for example, is conditionally stable.

✗