The Finite Element Method

Basics of FE

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We have to solve the following differential equation

$$-u'' = -\frac{d^2u}{dx^2} = f \quad \text{in }]0,1[\tag{1}$$

where

$$u(0) = 0 \quad u(1) = \alpha \tag{2}$$

Weak form

From (1) and (2),

$$A(u) = \frac{d^2u}{dx^2} + f \tag{3}$$

$$B(u) = 0 \tag{4}$$

Then,

$$\int_0^1 WA(u)\,\mathrm{d}x = 0 \tag{5}$$

$$\int_0^1 W\left(\frac{d^2u}{dx^2} + f\right)\mathrm{d}x = \int_0^1 W\frac{d^2u}{dx^2}\mathrm{d}x + \int_0^1 Wf\mathrm{d}x = 0 \tag{6}$$

which, after integrating by parts, becomes

$$W\frac{du}{dx}\bigg|_0^1 - \int_0^1 \frac{du}{dx}\frac{dW}{dx}\mathrm{d}x + \int_0^1 Wf\mathrm{d}x = 0 \tag{7}$$

Therefore, the weak form of the problem is:

$$\int_0^1 \frac{du}{dx}\frac{dW}{dx}\mathrm{d}x = \int_0^1 Wf\mathrm{d}x + W\frac{du}{dx}\bigg|_0^1 \tag{8}$$

where $u \simeq u^h = \sum_{j=0}^{n} a_j N_j$, known as the FE approximation. N_j are the so called shape functions, shown in Figure 1 for $j = 0, ..., 3$.

Linear system of equations

We proceed to discretize the weak form of the differential equation by using the above definition of u. For the weights W, we use the Galerkin procedure: $W \equiv W_i = N_i$. Then,

$$\int_0^1 \sum_j a_j \frac{dN_j}{dx}\frac{dN_i}{dx}\mathrm{d}x = \int_0^1 N_i f\mathrm{d}x + N_i\frac{du}{dx}\bigg|_0^1 \Rightarrow \sum_j a_j \int_0^1 \frac{dN_j}{dx}\frac{dN_i}{dx}\mathrm{d}x = \int_0^1 N_i f\mathrm{d}x + N_i\frac{du}{dx}\bigg|_0^1 \tag{9}$$

Now, defining $\int_0^1 \frac{dN_j}{dx}\frac{dN_i}{dx}\mathrm{d}x \equiv K_{ij}$, $\int_0^1 N_i f\mathrm{d}x \equiv f_i$, $N_i\frac{du}{dx}\Big|_0^1 \equiv q_i$, the last expression can be rewritten as:

$$\sum_j a_j K_{ij} = f_i + q_i \tag{10}$$

which is the linear system of equations that must be computed in order to solve the differential equation given.

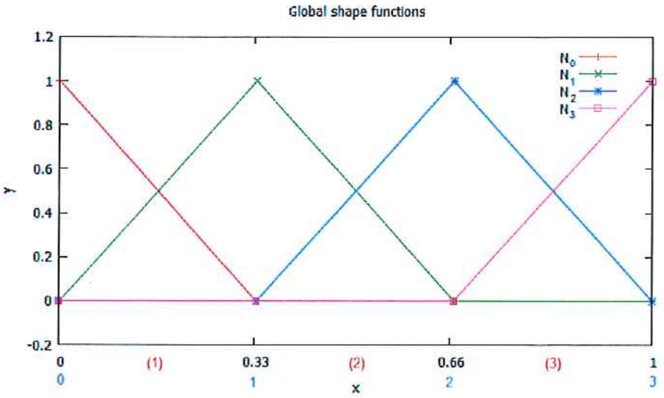

Figure 1: Global shape functions. Nodes are shown in blue, while each element is shown in red.

FE approximation

We now set (1) and (2) as:

$$-\frac{d^2u}{dx} = \sin x \tag{11}$$

$$u(0) = 0, u(1) = 3 \tag{12}$$

with $u \simeq u^h = \sum_{j=0}^{3} a_j N_j$

We will use local coordinates to solve the linear system of equations given in (10). This means that instead of working with the shape functions shown in Figure 1, we use the ones shown in the following Figure:

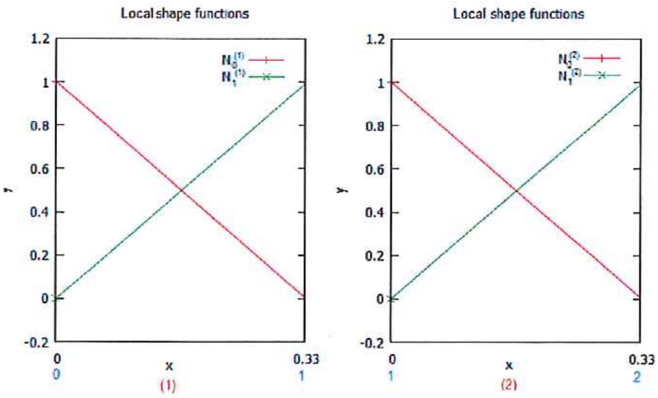

Figure 2: Local shape functions. Nodes are shown in blue, while each element is shown in red.

Therefore, for the first element,

$$K_{00}^{(1)} = \int_0^{1/3} \frac{dN_0^{(1)}}{dx}\frac{dN_0^{(1)}}{dx} \mathrm{d}x = \int_0^{1/3} 9\mathrm{d}x = 3 \tag{13}$$

$$K_{01}^{(1)} = \int_0^{1/3} \frac{dN_0^{(1)}}{dx}\frac{dN_1^{(1)}}{dx} \mathrm{d}x = -\int_0^{1/3} 9\mathrm{d}x = -3 = K_{10}^{(1)} \tag{14}$$

$$K_{11}^{(1)} = \int_0^{1/3} \frac{dN_1^{(1)}}{dx}\frac{dN_1^{(1)}}{dx} \mathrm{d}x = \int_0^{1/3} 9\mathrm{d}x = 3 \tag{15}$$

Since all the three elements have the same size and properties, $N_0^{(1)} = N_0^{(2)} = N_0^{(3)} \equiv -3x+1$, and $N_1^{(1)} = N_1^{(2)} = N_1^{(3)} \equiv 3x$. This means that:

$$K_{00}^{(1)} = K_{00}^{(2)} = K_{00}^{(3)} = 3 \tag{16}$$
$$K_{01}^{(1)} = K_{10}^{(1)} = K_{01}^{(2)} = K_{10}^{(2)} = K_{01}^{(3)} = K_{10}^{(3)} = -3 \tag{17}$$
$$K_{11}^{(1)} = K_{11}^{(2)} = K_{11}^{(3)} = 3 \tag{18}$$

We go on computing f_i,

$$f_0^{(1)} = \int_0^{1/3} N_0^{(1)} f \mathrm{d}x = \int_0^{1/3} (-3x+1)\sin x \mathrm{d}x = 1 - 3\sin\left(\frac{1}{3}\right) \tag{19}$$
$$f_1^{(1)} = \int_0^{1/3} N_1^{(1)} f \mathrm{d}x = \int_0^{1/3} 3x \sin x \mathrm{d}x = 3\sin\left(\frac{1}{3}\right) - \cos\left(\frac{1}{3}\right) \tag{20}$$
$$\tag{21}$$

And for similar reasons as before, $f_0^{(1)} = f_0^{(2)} = f_0^{(3)}$, $f_1^{(1)} = f_1^{(2)} = f_1^{(3)}$. Now, for q_i:

$$q_0^{(1)} = N_0^{(1)} \left.\frac{du^{(1)}}{dx}\right|_0^{1/3} = -\left.\frac{du^{(1)}}{dx}\right|_0 \equiv -\left.\frac{du}{dx}\right|_0 \tag{22}$$
$$q_1^{(1)} = N_1^{(1)} \left.\frac{du^{(1)}}{dx}\right|_0^{1/3} = \left.\frac{du^{(1)}}{dx}\right|_{\frac{1}{3}} \equiv \left.\frac{du}{dx}\right|_{\frac{1}{3}} \tag{23}$$
$$q_0^{(2)} = N_0^{(2)} \left.\frac{du^{(2)}}{dx}\right|_0^{1/3} = -\left.\frac{du^{(2)}}{dx}\right|_0 \equiv -\left.\frac{du}{dx}\right|_{\frac{1}{3}} \tag{24}$$
$$q_1^{(2)} = N_1^{(2)} \left.\frac{du^{(2)}}{dx}\right|_0^{1/3} = \left.\frac{du^{(2)}}{dx}\right|_{\frac{1}{3}} \equiv \left.\frac{du}{dx}\right|_{\frac{2}{3}} \tag{25}$$
$$q_0^{(3)} = N_0^{(3)} \left.\frac{du^{(3)}}{dx}\right|_0^{1/3} = -\left.\frac{du^{(3)}}{dx}\right|_0 \equiv -\left.\frac{du}{dx}\right|_{\frac{2}{3}} \tag{26}$$
$$q_1^{(3)} = N_1^{(3)} \left.\frac{du^{(3)}}{dx}\right|_0^{1/3} = \left.\frac{du^{(3)}}{dx}\right|_{\frac{1}{3}} \equiv \left.\frac{du}{dx}\right|_1 \tag{27}$$

where in the right-hand side of each q_i there is the expression in global coordinates. Notice that in global coordinates is easy to see that $q_0^{(2)} = -q_1^{(1)}$ and $q_0^{(3)} = -q_1^{(2)}$

Finally, the following matrix equation can be computed:

$$\begin{bmatrix} K_{00}^{(1)} & K_{01}^{(1)} & 0 & 0 \\ K_{10}^{(1)} & K_{11}^{(1)} + K_{00}^{(2)} & K_{01}^{(2)} & 0 \\ 0 & K_{10}^{(2)} & K_{11}^{(2)} + K_{00}^{(3)} & K_{01}^{(3)} \\ 0 & 0 & K_{10}^{(3)} & K_{11}^{(3)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_0^{(1)} + q_0^{(1)} \\ f_1^{(1)} + f_0^{(2)} + q_0^{(2)} + q_1^{(1)} \\ f_1^{(2)} + f_0^{(3)} + q_1^{(2)} + q_0^{(3)} \\ f_1^{(3)} + q_1^{(3)} \end{bmatrix} \tag{28}$$

which becomes,

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - 3\sin\left(\frac{1}{3}\right) - \left.\frac{du}{dx}\right|_0 \\ 1 - \cos\left(\frac{1}{3}\right) \\ 1 - \cos\left(\frac{1}{3}\right) \\ 3\sin\left(\frac{1}{3}\right) - \cos\left(\frac{1}{3}\right) + \left.\frac{du}{dx}\right|_1 \end{bmatrix} \tag{29}$$

It is easy to see that (29) yields to the following system of equations:

$$6a_1 - 3a_2 = -\cos\left(\frac{1}{3}\right) + 1 \tag{30}$$

$$-3a_1 + 6a_2 = -\cos\left(\frac{1}{3}\right) + 10 \tag{31}$$

Once it is solved,

$$a_1 = \frac{1}{3}\left(4 - \cos\left(\frac{1}{3}\right)\right) \approx 1.018348 \qquad a_2 = \frac{1}{3}\left(7 - \cos\left(\frac{1}{3}\right)\right) \approx 2.018347 \tag{32}$$

and then, replacing (32) into the other two equations of (29) (which arise from its first row and its last row) yields to:

$$\left.\frac{du}{dx}\right|_0 = 3a_1 + 1 - 3\sin\left(\frac{1}{3}\right) \approx 3.07346 \tag{33}$$

$$\left.\frac{du}{dx}\right|_1 = -3\sin\left(\frac{1}{3}\right) + \cos\left(\frac{1}{3}\right) + 9 - 3a_2 \approx 2.9083 \tag{34}$$

We finally compare these numerical results with the ones obtained from the analytical solution of the equation ($u(x) = \sin x + (3 - \sin 1)\, x$):

$$u\left(\frac{1}{3}\right) \approx 1.0467043 \simeq a_1 \qquad u\left(\frac{2}{3}\right) \approx 2.057389 \simeq a_2 \tag{35}$$

$$\left.\frac{du}{dx}\right|_0 = 3.158529 \simeq 3.07346 \qquad \left.\frac{du}{dx}\right|_1 \approx 2.698831 \simeq 2.9083 \tag{36}$$

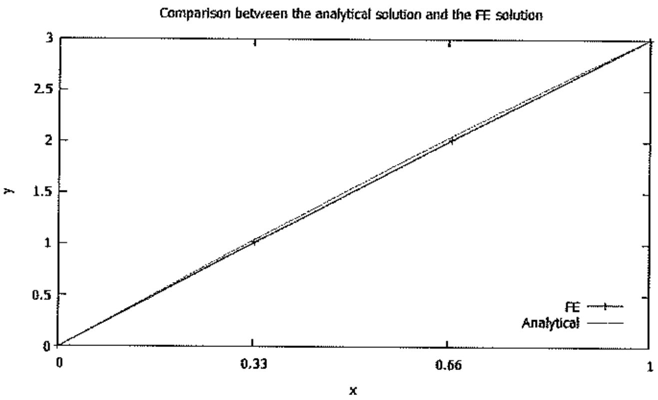

Figure 3: Comparison between $u \simeq u^h = \sum_{j=0}^{3} a_j N_j$ (FE solution) and $u(x) = \sin x + (3 - \sin 1)\, x$ (analytical solution).