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FINITE ELEMENTS January 20, 2014

Time allowed: 2:00 hours

All the exam should be developed on this sheets. No additional sheets will be corrected.

1. The Poisson equation is solved using the Finite Element Method in a rectangular domain of height H and width $3H$. The problem is stated as follows

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= x \quad \text{on } \Gamma_D \subset \partial\Omega \end{aligned}$$

where the source term f is constant and the Dirichlet boundary Γ_D includes the two lateral sides of the domain ($x = 0$ and $x = 3H$). In the rest of the boundary natural boundary conditions (homogeneous Neumann) are imposed. $\frac{du}{dy} = 0$

The mesh is constituted by 8 nodes and 6 triangular three-noded elements and it is characterized by the following nodal coordinates and connectivity:

$$\mathrm{X} = \begin{bmatrix} H & 0 \\ 2H & 0 \\ 2H & H \\ H & H \\ 0 & H \\ 0 & 0 \\ 3H & 0 \\ 3H & H \end{bmatrix} \quad \text{and } \mathrm{T} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 6 & 1 & 5 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 2 \\ 2 & 7 & 3 \\ 7 & 8 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

a) Represent graphically the mesh, numbering the elements and the nodes (both local and global node numbering).

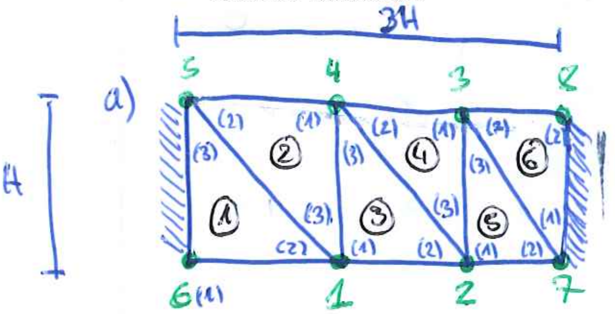
b) Compute the 3×3 elementary stiffness matrix for the first element (with vertices (0,0), (H,0) and (0,H)). Give the expressions of the shape functions and their derivatives.

c) Noting that all the elements in the mesh are similar to the first element, assemble the global 8×8 stiffness matrix without accounting for Dirichlet boundary conditions.

d) Repeat questions **b** and **c** for the force term vector.

e) Use the Dirichlet boundary conditions and find the reduced 4×4 linear system of equations to be solved.

a)



Local (i)
Global (i)
Element (i)

b)

$$\begin{array}{cccc} & 1 & 2 & 3 \text{ (Local)} \\ & 6 & 1 & 5 \text{ (Global)} \\ K_1 = & \left[\begin{matrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{matrix}\right. & & \left.\right] \end{array}$$

$$N_i^{(e)} = \frac{1}{2A^{(e)}}\left(a_i^{(e)} + b_i^{(e)} x + c_i^{(e)} y\right) \quad i = 1,2,3$$

$$2A^{(e)} = \begin{vmatrix} 1 & x_i^{(e)} & y_i^{(e)} \\ 1 & x_j^{(e)} & y_j^{(e)} \\ 1 & x_k^{(e)} & y_k^{(e)} \end{vmatrix} \quad \text{where;} \quad \left\{ \begin{array}{l} a_i^{(e)} = x_j^{(e)} y_k^{(e)} - x_k^{(e)} y_j^{(e)} \\ b_i^{(e)} = y_j^{(e)} - y_k^{(e)} \\ c_i^{(e)} = x_k^{(e)} - x_j^{(e)} \end{array}\right.$$

$$\frac{\partial N_i^{(e)}}{\partial x} = \frac{b_i}{2A^{(e)}} \; ; \quad \frac{\partial N_i^{(e)}}{\partial y} = \frac{c_i}{2A^{(e)}}$$

Finally

$$K_d^{(1)} = \frac{1}{4A^{(e)}} \cdot \begin{array}{c} \begin{array}{ccc} 6 & 1 & 5 \end{array} \\ \begin{bmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + b_2 b_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{bmatrix} \end{array}$$

where; $b_1 = -H \quad b_2 = H \quad b_3 = 0$
$c_1 = -H \quad c_2 = 0 \quad c_3 = H$

shape functions ..... derivatives
elemental K^e

$$A^{(e)} = \frac{b \cdot a}{2} = \frac{H \cdot H}{2} = \frac{H^2}{2} \; ;$$

c)

K_G

	1 2 / 1	3 4 / 2	5 6 / 3	7 8 / 4	9 10 / 5	11 12 / 6	13 14 / 7	15 16 / 8
1 2 / 1	$K_{22}^{(1)} + K_{33}^{(2)} + K_{11}^{(3)}$	$K_{12}^{(3)}$	0	$K_{31}^{(2)} + K_{13}^{(3)}$	$K_{23}^{(1)} + K_{32}^{(2)}$	$K_{21}^{(1)}$	0	0
3 4 / 2	·	$K_{11}^{(5)} + K_{22}^{(3)} + K_{33}^{(4)}$	$K_{31}^{(4)} + K_{13}^{(5)}$	$K_{23}^{(3)} + K_{32}^{(4)}$	0	0	$K_{12}^{(5)}$	0
5 6 / 3		·	$K_{11}^{(4)} + K_{33}^{(5)} + K_{33}^{(6)}$	$K_{12}^{(4)}$	0	0	$K_{32}^{(5)} + K_{31}^{(6)}$	$K_{32}^{(6)}$
7 8 / 4			·	$K_{11}^{(2)} + K_{33}^{(3)} + K_{22}^{(4)}$	$K_{12}^{(2)}$	0	0	0
9 10 / 5				·	$K_{22}^{(2)} + K_{33}^{(1)}$	$K_{31}^{(1)}$	0	0
11 12 / 6					·	$K_{12}^{(1)}$	0	0
13 14 / 7						·	$K_{22}^{(5)} + K_{11}^{(6)}$	$K_{12}^{(6)}$
15 16 / 8							·	$K_{22}^{(6)}$

SYMMETRIC

d) Force will be compute

$$f_1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \end{bmatrix} \begin{matrix} 6 \\ 1 \\ 5 \end{matrix} = \frac{Q A^{(e)}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{matrix} 6 \\ 1 \\ 5 \end{matrix}$$

$$A^{(e)} = \frac{b \cdot a}{2} = \frac{H^2}{2}$$

Q = cte source term.

$$f_2 = \begin{bmatrix} \cancel{1} \\ 2 \\ 3 \end{bmatrix} \begin{matrix} 4 \\ 5 \\ 2 \end{matrix} \quad f_3 = \begin{bmatrix} \cancel{} \\ \\ \end{bmatrix} \begin{matrix} 1 \\ 5 \\ 4 \end{matrix} \quad f_6 = \begin{bmatrix} \frac{1}{6} \\ \frac{2}{6} \\ \frac{3}{6} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 3 \end{matrix}$$

$$f_4 = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 2 \end{matrix} \quad f_5 = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} \begin{matrix} 2 \\ 7 \\ 3 \end{matrix}$$

$$f_G = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} f_1^3 + f_3^2 + f_2^1 \\ f_2^3 + f_1^5 + f_4^3 + f_5^1 \\ f_4^1 + f_5^3 + f_6^3 \\ f_2^1 + f_3^2 + f_4^2 \\ f_2^2 + f_3^1 \\ f_1^1 \\ f_5^2 + f_6^1 \\ f_6^2 \end{bmatrix}$$

e)

Vector of unknowns ?

$$a = \begin{array}{c} 1 \\ \cancel{2} \\ 3 \\ \cancel{4} \\ 5 \\ \cancel{6} \\ 7 \\ \cancel{8} \\ \cancel{9} \\ \cancel{10} \\ \cancel{11} \\ \cancel{12} \\ \cancel{13} \\ \cancel{14} \\ \cancel{15} \\ \cancel{16} \end{array} \begin{bmatrix} x_1 \\ \cancel{y_1} \; ① \\ x_2 \\ \cancel{y_2} \; ① \\ x_3 \\ \cancel{y_3} \; ① \\ x_4 \\ \cancel{y_4} \; ① \\ \cancel{x_5} \; ② \\ \cancel{y_5} \; ① \\ \cancel{x_6} \; ② \\ \cancel{y_6} \; ① \\ \cancel{x_7} \; ② \\ \cancel{y_7} \; ① \\ \cancel{x_8} \; ② \\ \cancel{y_8} \; ① \end{bmatrix}$$

② Direcht Boundary condition

① Neuman Boundary condition.

→ We need to solve lines 1, 3, 5 and 7.

$$K_G = \begin{array}{c|cccc} & 1 & 3 & 5 & 7 \\ \hline 1 & K_{22}^{(1)} + K_{33}^{(2)} + K_{11}^{(3)} & 0 & K_{23}^{(1)} + K_{32}^{(2)} & 0 \\ 3 & \cdot & K_{11}^{(4)} + K_{33}^{(5)} + K_{33}^{(6)} & 0 & K_{32}^{(5)} + K_{31}^{(6)} \\ 5 & \cdot & \cdot & K_{22}^{(2)} + K_{33}^{(1)} & 0 \\ 7 & \cdot & \cdot & \cdot & K_{22}^{(5)} + K_{11}^{(6)} \end{array}$$

SYMMETRIC

2. The following ODE has to be solved using the Finite Element Method

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$$\begin{aligned} -u'' + u &= f && \text{in }]0,1[\\ u(0) &= 0 && \text{at } x = 0 \\ u'(1) &= \alpha && \text{at } x = 1 \end{aligned}$$

with a uniform discretization $\{x_0, x_1, x_3, x_4, x_4\}$, with $x_i = i/4$, for $i = 0, 1, \ldots, 4$, and both a mesh of 4 linear elements and a mesh of 2 quadratic elements.

a) Find the weak form of the problem.

b) Obtain the general expression of the elementary matrices for linear and quadratic elements.

c) Assemble the global matrices in the two cases.

a) $-u'' + u = f$; $\xrightarrow{\cdot\,\mathrm{WRX}}$ $-\int_0^1 u''\cdot w\,dx + \int_0^1 w\cdot u\,dx = \int_0^1 w f\,dx$;

Applying $\left[u\cdot v - \int v\,du \;\; \begin{matrix} dv = u'' & v = u' \\ u = w & du = w' \end{matrix}\right]$; $-w\cdot u'\Big|_0^1 + \int_0^1 u' w'\,dx + \int_0^1 w\cdot u\,dx = \int_0^1 w f\,dx$; $\begin{cases} w(0) = 0 \\ w(1) = 0 \end{cases}$

① $\left[\int_0^1 u' w'\,dx = \int_0^1 w\cdot f\,dx + w\cdot u'\Big|_0^1 + \cancel{\int_0^1 w\cdot u\,dx}\right]$ weak form.

b) Applying $\left[u \approx u^h = \sum_{j=1}^{n} N_j\cdot u_j\right]$ and Galerkin $[w_i = N_i]$; to ①

$$\int_0^1 N_i'\left(\sum_{j=1}^{n} N_j'\cdot u_j\right)dx = \int_0^1 N_i\cdot f\,dx + N_i(1)\cdot \cancel{u'(1)}^{\alpha} - N_i(0)\cdot u'(0) \;;$$

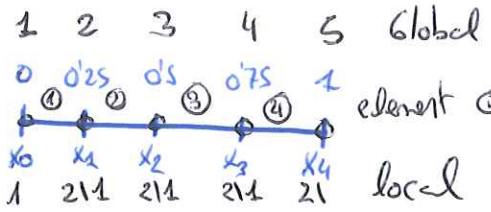
$$\left[\underbrace{\sum_{j=1}^{n}\int_0^1 N_i'\cdot N_j'}_{K_{ij}} \cdot \underbrace{u_j}_{a}\,dx = \underbrace{\int_0^1 N_i\cdot f\,dx}_{f} + \underbrace{\alpha N_i(1)}_{q} - \cancel{N_i(0)}^{\,w(0)=0}\cdot u'(0) \;;\right]$$

$$K_{ij} \cdot a = f + q$$

→ We could use this expression for linear and quadratic elements the difference will be in the number of nodes and the shape functions used in each case.

c)

- Linear element (4)



$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix}$$

$$K_1 = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & K_{11} & K_{12} \\ 2 & K_{21} & K_{22} \end{array} \qquad K_2 = \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & K_{11} & K_{12} \\ 3 & K_{21} & K_{22} \end{array}$$

$$K_3 = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & K_{11} & K_{12} \\ 4 & K_{21} & K_{22} \end{array} \qquad K_4 = \begin{array}{c|cc} & 4 & 5 \\ \hline 4 & K_{11} & K_{12} \\ 5 & K_{21} & K_{22} \end{array}$$

$$K_G = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 \\ 2 & K_{21}^{(1)} & K_{22}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 \\ 3 & 0 & K_{21}^{(2)} & K_{22}^{(2)}+K_{11}^{(3)} & K_{12}^{(3)} & 0 \\ 4 & 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)}+K_{11}^{(4)} & K_{12}^{(4)} \\ 5 & 0 & 0 & 0 & K_{21}^{(4)} & K_{22}^{(4)} \end{array}$$

$$f_i = \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} f_i^1 \\ f_i^2 \end{bmatrix}$$ $i = 1, \dots, 4.$ (each element) $\rightarrow$ $$f_G = \begin{bmatrix} f_1^1 \\ f_1^2 + f_2^1 \\ f_2^2 + f_3^1 \\ f_3^2 + f_4^1 \\ f_4^2 \end{bmatrix}$$

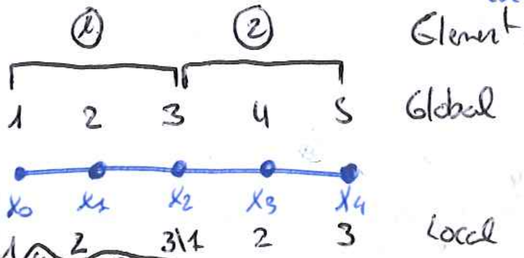
$$f_1 = \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} f_1^1 \\ f_1^2 \end{bmatrix} \quad f_2 = \begin{array}{c} 2 \\ 3 \end{array}\begin{bmatrix} f_2^1 \\ f_2^2 \end{bmatrix} \quad f_3 = \begin{array}{c} 3 \\ 4 \end{array}\begin{bmatrix} f_3^1 \\ f_3^2 \end{bmatrix} \quad f_4 = \begin{array}{c} 4 \\ 5 \end{array}\begin{bmatrix} f_4^1 \\ f_4^2 \end{bmatrix}$$

$$q_i = \begin{bmatrix} q_i^1 \\ q_i^2 \end{bmatrix} \rightarrow q_G = \begin{bmatrix} q_1^1 \\ q_1^2 + q_2^1 \\ q_2^2 + q_3^1 \\ q_3^2 + q_4^1 \\ q_4^2 \end{bmatrix}$$

$i = 1, \dots, 4,$

Final assembly is just + each line.

- Quadratic element (2)



$$K_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & K_{11} & K_{12} & K_{13} \\ 2 & K_{21} & K_{22} & K_{23} \\ 3 & K_{31} & K_{32} & K_{33} \end{array} \qquad K_2 = \begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & K_{11} & K_{12} & K_{13} \\ 4 & K_{21} & K_{22} & K_{23} \\ 5 & K_{31} & K_{32} & K_{33} \end{array}$$

$$K_G = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & K_{11}^{(1)} & K_{12} & K_{13}^{(1)} & 0 & 0 \\ 2 & K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ 3 & K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ 4 & 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 5 & 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(1)} \end{array}$$

$$f_i = \begin{bmatrix} f_i^1 \\ f_i^2 \\ f_i^3 \end{bmatrix}$$ $i = 1, 2.$ $$f_G = \begin{bmatrix} f_1^1 \\ f_1^2 \\ f_1^3 + f_2^1 \\ f_2^2 \\ f_2^3 \end{bmatrix}$$

Finally + each line.

$$q_i = \begin{bmatrix} q_i^1 \\ q_i^2 \\ q_i^3 \end{bmatrix} \rightarrow q_G = \begin{bmatrix} q_1^1 \\ q_1^2 \\ q_1^3 + q_2^1 \\ q_2^2 \\ q_2^3 \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

$$\boxed{K \cdot a = f + q}$$

3. COMMENT ON THE APPROPRIATENESS OF EACH OF THE FOLLOWING STATEMENTS, DISCUSSING WHICH PARTS ARE TRUE AND WHICH ARE FALSE.

a) The finite element approximation in the linear elastic problem in 1D should satisfy certain conditions which guarantee that as the mesh is refined, the numerical solution converges to the exact one. These are:

1. The continuity condition: the displacement should have C0 continuity within each element and along the element interfaces.

~~True? C^0 continuity indicates that just the function is continuous.~~

True. Displacement field for C^0 elements, or its first derivative field for C^1 elements, must be continuous along interelemental boundaries. when integrality exactly always converge. ?

2. The derivability condition: the derivatives of the function approximating the displacement should exist up to the order of the derivatives appearing in the element integrals.

This requires that the shape functions should be at least 1 order poly.

~~False? Exple: ϕ of order 2 we just need C^1 continuous function and first derivate~~ True. For instance, the axially loaded rod problem the element expressions derived from PVW contain 1 order derivat.

3. The integrability condition: the integrals appearing in the element expressions must have a primitive function. If *m-th* order derivative of the displacement field appear in the weak form, the shape functions must be C^{m+1}

False. If mth order derivatives of the displacement field appears in the PVW, the displacement must be C^{m-1} continuous.

4. The rigid body condition: when a rigid body motion is imposed, no strain should occur in the element. That is satisfied for a single element is the sum of the shape functions derivatives at any point is equal to 1

True and false. True until \\; then considering simpled 2noded axially loaded rod element with equal prescribed displacement $\bar{u}$ $u = N_1\bar{u} + N_2\bar{u} = (N_1+N_2)\bar{u}$ for $u=\bar{u}$ $[N_1+N_2=1]$ not the derivatives.

5. The constant strain condition: the displacement function has to be such that is nodal displacements are compatible with a constant strain field, such constant strain should be obtained. This condition is incorporated by the rigid body condition.

True. As elements get smaller, nearly constant strain conditions will prevail in them.

b) About error estimates.

1. A priori error estimates are not well suited to compute the level of error of the FE approximation.

2. On the contrary, a posteriori error estimates deliver an upper-bound of the actual error, provided it is measured with the energy norm.

3. If the user prefers other measures of the error (different than the energy norm), a posteriori error estimates cannot be used anymore.

c) In the context of Structural Dynamics.

1. The modal approach is always preferred to the direct time integration because the number of d.o.f. is drastically reduced.

2. The stability of the Newmark method is guaranteed independently of the selected Δt.

3. In modal analysis, the convergence ratio is different for every eigen-mode, being better for those associated with lower eigen-frequencies