6-9

**FINITE ELEMENTS** January 20, 2014

Time allowed: 2:00 hours

**All the exam should be developed on this sheets. No additional sheets will be corrected.**

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1. The Poisson equation is solved using the Finite Element Method in a rectangular domain of height $H$ and width $3H$. The problem is stated as follows

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= x \quad \text{on } \Gamma_D \subset \partial\Omega \end{aligned}$$

where the source term $f$ is constant and the Dirichlet boundary $\Gamma_D$ includes the two lateral sides of the domain ($x = 0$ and $x = 3H$). In the rest of the boundary natural boundary conditions (homogeneous Neumann) are imposed.

The mesh is constituted by 8 nodes and 6 triangular three-noded elements and it is characterized by the following nodal coordinates and connectivity:

$$\mathtt{X} = \begin{bmatrix} H & 0 \\ 2H & 0 \\ 2H & H \\ H & H \\ 0 & H \\ 0 & 0 \\ 3H & 0 \\ 3H & H \end{bmatrix} \quad \text{and } \mathtt{T} = \begin{bmatrix} 6 & 1 & 5 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 2 \\ 2 & 7 & 3 \\ 7 & 8 & 3 \end{bmatrix}$$

**a)** Represent graphically the mesh, numbering the elements and the nodes (both local and global node numbering).

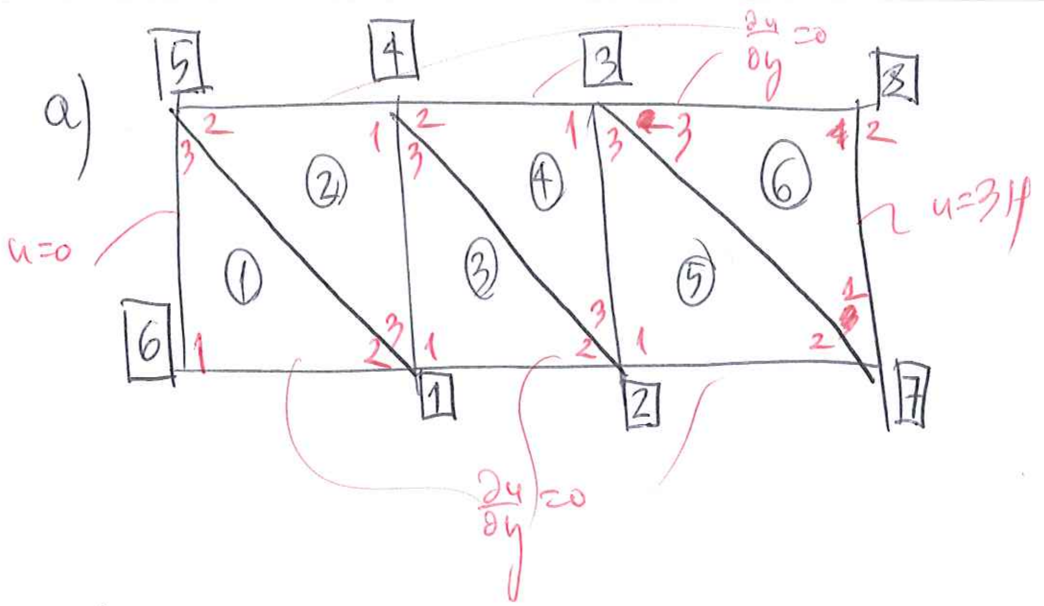
**b)** Compute the 3×3 elementary stiffness matrix for the first element (with vertices (0,0), (H,0) and (0,H)). Give the expressions of the shape functions and their derivatives.

**c)** Noting that all the elements in the mesh are similar to the first element, assemble the global 8×8 stiffness matrix without accounting for Dirichlet boundary conditions.

**d)** Repeat questions **b** and **c** for the force term vector.

**e)** Use the Dirichlet boundary conditions and find the reduced 4×4 linear system of equations to be solved.

a)




b) Element 1:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -H \\ H \\ 0 \end{pmatrix}; \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -H \\ 0 \\ H \end{pmatrix};$$

Element 2:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} H \\ -H \\ 0 \end{pmatrix}; \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} H \\ 0 \\ -H \end{pmatrix}$$

$$\underset{\sim}{K}^{(1)} = \frac{1}{4\frac{H^2}{2}} \begin{pmatrix} 2H^2 & -H^2 & -H^2 \\ -H^2 & H^2 & 0 \\ -H^2 & 0 & H^2 \end{pmatrix} = \begin{pmatrix} K^1_{11} & K^1_{12} & K^1_{13} \\ & K^1_{22} & K^1_{23} \\ (\text{Sym}) & & K^1_{33} \end{pmatrix}$$

$$\underset{\sim}{K}^{(2)} = \frac{1}{4\frac{H^2}{2}} \begin{pmatrix} 2H^2 & -H^2 & -H^2 \\ -H^2 & H^2 & 0 \\ -H^2 & 0 & H^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \underset{\sim}{K}^{(1)}$$

Shape functions: $N_i = \frac{1}{H^2}(a_i + b_i x + c_i y)$ ; $\frac{\partial N_i}{\partial x} = \frac{b_i}{H^2}$ ; $\frac{\partial N_i}{\partial y} = \frac{c_i}{H^2}$

$N_1, N_2, N_3 \ldots$ particularized

$\frac{\partial N_i}{\partial N_j} \ldots$

So, all the $\underset{\sim}{K}^{(i)}$ are the same. $i = 1, \ldots, 6$.

c)

$$\underset{\sim}{K}_{8\times 8} = \begin{pmatrix} k^1_{22}+k^2_{33}+k^3_{11} & & & & & & & \\ k^3_{12} & k^3_{22}+k^4_{33}+k^5_{11} & & & & & & \\ 0 & k^4_{31}+k^5_{31} & k^4_{11}+k^5_{33}+k^6_{33} & & & (\text{SYMM.}) & & \\ k^2_{31}+k^3_{31} & k^4_{32}+k^3_{32} & k^4_{21} & k^2_{11}+k^3_{33}+k^4_{22} & & & & \\ k^1_{31}+k^2_{32} & 0 & 0 & k^2_{21} & k^1_{33}+k^2_{22} & & & \\ k^1_{12} & 0 & 0 & 0 & k^1_{13} & k^1_{11} & & \\ 0 & k^5_{12} & k^5_{23}+k^6_{31} & 0 & 0 & 0 & k^5_{22}+k^6_{11} & \\ 0 & 0 & k^6_{23} & 0 & 0 & 0 & k^6_{12} & k^6_{22} \end{pmatrix}$$

assembled

d) $$\bar{f}_f = \frac{f\,H^2/2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{fH^2}{6}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

assembled : $$\bar{f}_{8\times 1} = \begin{pmatrix} f^1_2+f^2_3+f^3_1 \\ f^3_2+f^4_3+f^5_1 \\ f^4_1+f^5_3+f^6_3 \\ f^2_1+f^3_3+f^4_2 \\ f^1_3+f^2_2 \\ f^1_1 \cancel{+f^6} \\ f^5_2+f^6_1 \\ f^6_2 \end{pmatrix} = \frac{fH^2}{6}\begin{pmatrix}3\\3\\3\\3\\2\\1\\2\\1\end{pmatrix}$$

e) The only unknowns are $\phi_1, \phi_2, \phi_3, \phi_4$.
$\phi_5 = \phi_6 = 0$ and $\phi_7 = \phi_8 = 3H$. So we can suppress their files and columns and pass their contribution to the RHS:

$$\underbrace{\begin{pmatrix} \text{First } 4\times4 \text{ block} \\ \text{of the original } \underset{\sim}{K} \end{pmatrix}}_{\underset{\sim}{K}^*} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \underbrace{\begin{pmatrix} \text{First} \\ 4\times1 \\ \text{block} \\ \text{of} \\ \text{the original} \\ \bar{f} \end{pmatrix}}_{\underset{\sim}{f}^*} - \begin{pmatrix} 0 \\ K_{12}^5 \\ K_{23}^5 + K_{31}^6 + K_{23}^6 \\ 0 \end{pmatrix} \cdot 3H$$

That way we can obtain the $\phi_1, \phi_2, \phi_3$ and $\phi_4$ values

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2. The following ODE has to be solved using the Finite Element Method

$$\begin{aligned} -u'' + u &= f && \text{in } ]0,1[ \\ u(0) &= 0 && \text{at } x = 0 \\ u'(1) &= \alpha && \text{at } x = 1 \end{aligned}$$

with a uniform discretization $\{x_0, x_1, x_3, x_4, x_4\}$, with $x_i = i/4$, for $i = 0, 1, \ldots, 4$, and both a mesh of 4 linear elements and a mesh of 2 quadratic elements.

a) Find the weak form of the problem.

b) Obtain the general expression of the elementary matrices for linear and quadratic elements.

c) Assemble the global matrices in the two cases.

a) $\displaystyle\int_0^1 w(u'' - u + f) + \left[\bar{w}(u' - \alpha)\right]_1 = 0$

$\displaystyle\int_0^1 w u'' = [w \cdot u']_0^1 - \int_0^1 w'u' - \int_0^1 wu + \int_0^1 wf + \bar{w}u'\Big|_1 - \bar{w}\alpha\Big|_1 = 0$

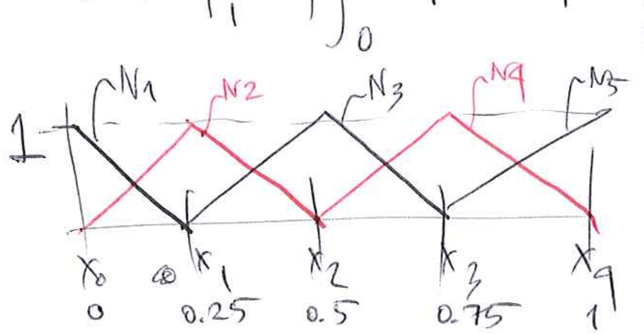
We take $\bar{w} = -w$

$\Rightarrow$ $\displaystyle -wu'\Big|_0 - \int_0^1 w'u' - \int_0^1 wu + \int_0^1 wf \cancel{-wu'\Big|_1} + w\alpha\Big|_1 \cancel{+wu'\Big|_1} = 0$

$\displaystyle\underbrace{\int_0^1 (w'u' + wu)}_{\tilde{K}\cdot a} = \underbrace{\int_0^1 wf \cancel{-wu'\Big|_0} + w\alpha\Big|_1}_{\bar{f}}$

b) So $\displaystyle K_{ij} = \int_0^1 N_i' N_j' + N_i N_j$

and $\displaystyle f_i = f\int_0^1 N_i - N_i u'\Big|_0 + N_i\alpha\Big|_1$




$N_1 = 1 - 4x$

$N_2 = \begin{cases} 4x & (0, 0.25) \\ 2 - 4x & (0.25, 0.5) \end{cases}$

$N_3 = \begin{cases} -1 + 4x & (0.25, 0.5) \\ 3 - 4x & (0.5, 0.75) \end{cases}$

$N_4 = \begin{cases} -2 + 4x & \text{in } (0.5, 0.75) \\ 4 - 4x & \text{in } (0.75, 1) \end{cases}$

$N_5 = -3 + 4x$ $(0.75, 1)$

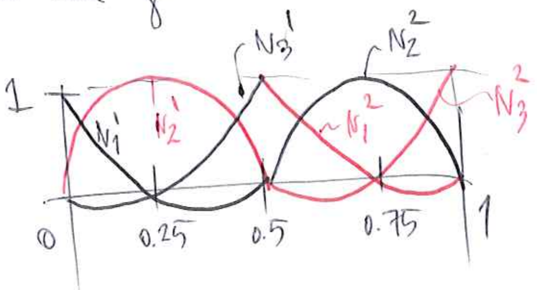
For the linear, we obtain: taking $\boxed{f = \alpha = 1 \quad \text{and} \quad u'(0) = K}$

$$\underset{\sim}{K} = \begin{pmatrix} 49/12 & -3.958 & & & 0 \\ -3.958 & \cancel{49}/6 & -3.958 & & \\ & -3.958 & 49/6 & -3.958 & \\ & & -3.958 & 49/6 & -3.958 \\ 0 & & & -3.958 & 49/12 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0.125 - K \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$$

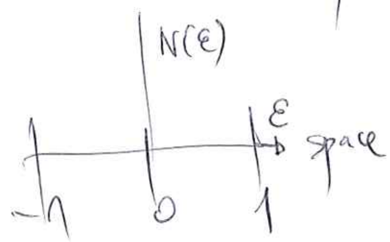
$\Rightarrow \bar{a} = \underset{\sim}{K}^{-1} \cdot \bar{f} \Rightarrow \bar{a} = \bar{a}(K) \Rightarrow$ imposing $u'(0) = K$

$\Rightarrow \boxed{K = 1.41} \Rightarrow \boxed{\underline{a} = (0,\ 0.325,\ 0.607,\ 0.865,\ 1.114)}$

For the quadratic




General form in a




$$\begin{cases} N_1 = \frac{1}{2}\varepsilon(\varepsilon - 1) \\ N_2 = (1 - \varepsilon^2) \\ N_3 = \frac{1}{2}\varepsilon(1 + \varepsilon) \end{cases}$$

We follow the same procedure as before: $K_{ij} = \int_0^1 N_i' N_j' + N_i N_j$

Element 1: $\underset{\sim}{K}^{(1)}, \ \bar{f}^{(1)}$

Element 2: $\underset{\sim}{K}^{(2)}, \ \bar{f}^{(2)} \Rightarrow \underset{\sim}{K}_{total} = \begin{pmatrix} \underset{\sim}{K}^1 & \bar{0} \\ \bar{0} & \underset{\sim}{K}^2 \end{pmatrix}; \ \bar{f}_{total} = \begin{pmatrix} \bar{f}^1 \\ \bar{f}^2 \end{pmatrix}$

$3\times3 \qquad 3\times1 \qquad\qquad 5\times5 \qquad\qquad 5\times1$

3. COMMENT ON THE APPROPRIATENESS OF EACH OF THE FOLLOWING STATEMENTS, DISCUSSING WHICH PARTS ARE TRUE AND WHICH ARE FALSE.

a) The finite element approximation in the linear elastic problem in 1D should satisfy certain conditions which guarantee that as the mesh is refined, the numerical solution converges to the exact one. These are:

1. The continuity condition: the displacement should have C0 continuity within each element and along the element interfaces.

It's false. Displacement must be continuous within each element. It is satisfied using polynomials aproximations. But, referring to the element interface, elements should be compatible or conforming.

2. The derivability condition: the derivatives of the function approximating the displacement should exist up to the order of the derivatives appearing in the element integrals.

That is true. For example, in an axially loaded rod the element expressions derived from the PVW contain first order derivaties only. This requires that the shape functions to be at least 1rst order.

3. The integrability condition: the integrals appearing in the element expressions must have a primitive function. If *m-th* order derivative of the displacement field appear in the weak form, the shape functions must be $C^{m+1}$

That is false. The shape functions shoud be $C^{m-1}$ continuous.

4. The rigid body condition: when a rigid body motion is imposed, no strain should occur in the element. That is satisfied for a single element is the sum of the shape functions derivatives at any point is equal to 1

That is false. It is the sum of the shape functions, not their derivatives.

5. The constant strain condition: the displacement function has to be such that is nodal displacements are compatible with a constant strain field, such constant strain should be obtained. This condition is incorporated by the rigid body condition.

That is true. As element get smaller, nearly constant strain condition will prevail in them. So the finite size element should be able to reproduce that constant strain.

b) About error estimates.

1. A priori error estimates are not well suited to compute the level of error of the FE approximation.

That is true, the a posteriori estimation methodologies are better. ✓

2. On the contrary, a posteriori error estimates deliver an upper-bound of the actual error, provided it is measured with the energy norm.

It is true that estimates an upper-bound of the error, but it is not necessary to measure it with the energy norm as the only possibility. ~

3. If the user prefers other measures of the error (different than the energy norm), a posteriori error estimates cannot be used anymore.

That is false. There are other measures of the error available. ✓

c) In the context of Structural Dynamics.

1. The modal approach is always preferred to the direct time integration because the number of d.o.f. is drastically reduced.

That is false. One of the useful things about the modal approach is that it allows to set up a system of decoupled ODEs. ✓

2. The stability of the Newmark method is guaranteed independently of the selected $\Delta t$.

That is false. The stability condition is $\left(\max_i |\lambda_i|\right)\Delta t \leq R_{crit}$ ✓

3. In modal analysis, the convergence ratio is different for every eigen-mode, being better for those associated with lower eigen-frequencies

That is false. It is better for higher eigen-frequencies.

✗