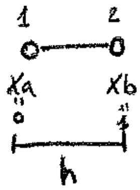
Homework 1

$$\begin{cases} -u'' = f & \text{in } [0,1] \\ u(0) = 0 \\ u(1) = \alpha \end{cases}$$

2 noded linear mesh



$x_i = ih$

$n = 1$

$h = \frac{1-0}{n} = \frac{1-0}{1} = 1$

1. Find the weak form of the problem. Describe the FE approximation u^h.

__WRF:__

$$\underbrace{-\int_0^1 w \cdot u'' \, dx}_{①} = \int_0^1 w f \, dx$$

① $dv = u''$ $\quad v = u'$ $\quad \int u\,dv = uv - \int v\,du$

$u = w$ $\quad du = w'$

$$-w \cdot u' \Big|_0^1 + \int_0^1 u' \cdot w' \, dx = \int_0^1 w \cdot f \, dx \ ;$$

② $$; \left[\int_0^1 u' \cdot w' \, dx = \int_0^1 w \cdot f \, dx + w \cdot u' \Big|_0^1 \right]$$ Weak form

Apply: ③ $\left[u \simeq u^h = \sum_{j=1}^{2} N_j \cdot u_j \right]$ and Galerkin ④ $\left[w_i = N_i \right]$ to ②;

$$; \int_0^1 \left(\sum_{j=1}^{2} u_j N_j' \right) \cdot N_i' \, dx = \int_0^1 N_i \cdot f \, dx + N_i \cdot u' \Big|_0^1 \ ;$$

$$; \quad \sum_{j=1}^{2} \underbrace{\int_0^1 (N_j' \cdot N_i')}_{k_{ij}} \underbrace{u_j}_{a_j} dx = \underbrace{\int_0^1 N_i \cdot f \, dx}_{f_i} + \underbrace{N_i(1) \cdot u'(1) - N_i(0) \cdot u'(0)}_{q_i}$$

2. Describe the linear system of equations to be solved.

From 1. section we identify:

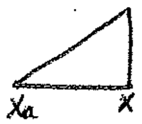
$$\left[k_{ij} = \int_{x_a}^{x_b} N_j' N_i' \, dx \right] \left[f_i = \int_{x_a}^{x_b} N_i \cdot f \, dx \right]$$

$$\left[q_i = N_i \cdot u' \Big|_{x_a}^{x_b} \right]$$

* $i, j = 1 \ldots 2$; apply linear shape functions: $N_1 = \frac{x_b - x}{h}$

$$\left[k_{11} = \int_{x_a}^{x_b} N_1' \cdot N_1' \, dx = \int_{x_a}^{x_b} -\frac{1}{h} \cdot -\frac{1}{h} \, dx = \frac{1}{h^2} \int_{x_a}^{x_b} dx = \frac{1}{h^2} \cdot \left(x \Big|_{x_a}^{x_b} \right) \right]$$

$N_2 = \frac{x - x_a}{h}$



For Galerkin

$$\left[k_{12} = k_{21} = \int_{x_a}^{x_b} -\frac{1}{h} \cdot \frac{1}{h} \, dx = -\frac{1}{h^2} \int_{x_a}^{x_b} dx = -\frac{1}{h^2} \left(x \cdot \Big|_{x_a}^{x_b} \right) \right]$$

$$\frac{dN_1}{dx} = -\frac{1}{h} \quad ; \quad \frac{dN_2}{dx} = \frac{1}{h}$$

$$\left[k_{22} = \int_{x_a}^{x_b} \frac{1}{h} \cdot \frac{1}{h} \, dx = \frac{1}{h^2} \left(x \Big|_{x_a}^{x_b} \right) \right]$$

* f_1, f_2 computed with $f(x) = \sin x$.

$$\left[f_1 = \int_{x_a}^{x_b} \left(\frac{x_b - x}{h} \right) \cdot \sin x \, dx = \int_{x_a}^{x_b} \frac{x_b}{h} \cdot \sin x \, dx - \frac{1}{h} \underbrace{\int_{x_a}^{x_b} x \cdot \sin x \, dx}_{\int u \, dv} ; \right.$$

②

$$; = \frac{x_b}{h} \cdot -\cos x \Big|_{x_a}^{x_b} - \frac{1}{h}\left(x \cdot \cos x \Big|_{x_a}^{x_b} + \int_{x_a}^{x_b} \cos x \, dx \right) =$$

$$= \frac{x_b}{h} \cdot -\cos x \Big|_{x_a}^{x_b} - \frac{1}{h} \cdot \left(x_b - \cos x_b - x_a \cdot \cos x_a + \sin x_b - \sin x_a \right) =$$

$$= \frac{1}{h}\left(x_b \cdot \cos x_a - x_a \cdot \cos x_a - \sin x_b + \sin x_a \right) \Big]$$

$$\Big[f_2 = \int_{x_a}^{x_b} \left(\frac{x - x_a}{h} \right) \cdot \sin x \, dx = \frac{1}{h} \left(\int_{x_a}^{x_b} x \cdot \sin x \, dx - x_a \int_{x_a}^{x_b} \sin x \, dx \right) =$$

$$= \frac{1}{h}\left(-x_b \cdot \cos x_b + \cancel{x_a \cdot \cos x_a} + \operatorname{sen} x_b - \operatorname{sen} x_a + x_a \cdot \cos x_b - \cancel{x_a \cdot \cos x_a} \right) =$$

$$= \frac{1}{h}\left(-x_b \cdot \cos x_b + \operatorname{sen} x_b - \operatorname{sen} x_a + x_a \cdot \cos x_b \right) \Big]$$

$$\Big[q_1^{(i)} = N_1(x_{bi})\, u'(x_{bi}) - N_1(x_{ai}) \cdot u'(x_{ai}) = \cancel{\frac{x_{bi} - x_{bi}}{h}}^{\,0} \cdot u'(x_{bi}) - \frac{x_{bi} - x_{ai}}{h} \cdot u'(x_{ai}) \Big]$$

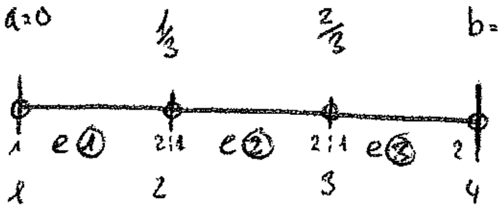
$$\Big[q_2^{(i)} = N_2(x_{bi})\, u'(x_{bi}) - N_2(x_{ai}) \cdot u'(x_{ai}) = \frac{x_{bi} - x_{ai}}{h} \cdot u'(x_{bi}) - \cancel{\frac{x_{ai} - x_{ai}}{h}}^{\,0} u'(x_{ai}) \Big]$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 + q_1 \\ f_2 + q_2 \end{bmatrix}$$

3: Compute the FE approximation u^h for $n=3$ $f(x)=\sin x$ and $q=3$.

Compare it with the exact solution $u(x)=\sin x+(3-\sin 1)\,x$.

$h=\frac{1}{n}=\frac{1}{3}$



Computing k_{ij}, f_i, q_i of ② for each element, he obtain:

- **Element 1** $(x_a=0,\ x_b=\frac{1}{3},\ h=\frac{1}{3})$

$k_{11}^{(1)}=3$ $\qquad$ $f_1^{(1)}=1-3\sin\left(\frac{1}{3}\right)$ $\qquad$ $q_1^{(1)}=-u'(0)$

$k_{12}=k_{21}^{(1)}=-3$ $\qquad$ $f_2^{(1)}=-\cos\left(\frac{1}{3}\right)+3\sin\left(\frac{1}{3}\right)$ $\qquad$ $q_2^{(1)}=u'\left(\frac{1}{3}\right)$

$k_{22}^{(1)}=3$

- **Element 2** $(x_a=\frac{1}{3},\ x_b=\frac{2}{3},\ h=\frac{1}{3})$

$k_{11}^{(2)}=3$ $\qquad$ $f_1^{(2)}=\cos\left(\frac{1}{3}\right)-3\sin\left(\frac{2}{3}\right)+3\sin\left(\frac{1}{3}\right)$

$k_{12}=k_{21}^{(2)}=-3$ $\qquad$ $f_2^{(2)}=-\cos\left(\frac{2}{3}\right)+3\sin\left(\frac{2}{3}\right)-3\sin\left(\frac{1}{3}\right)$

$k_{22}^{(2)}=3$

$q_1^{(2)}=-u'\left(\frac{1}{3}\right)$

$q_2^{(2)}=u'\left(\frac{2}{3}\right)$

- Element 3 ($x_a = \frac{2}{3}$, $x_b = 1$, $h = \frac{1}{3}$)

$$k_{11}^{(3)} = 3 \qquad f_1^{(3)} = \cos\left(\frac{2}{3}\right) - 3\sin(1) + 3\sin\left(\frac{2}{3}\right)$$

$$k_{12}^{(3)} = k_{21}^{(3)} = -3 \qquad f_2^{(3)} = -\cos(1) + 3\sin(1) - 3\sin\left(\frac{2}{3}\right)$$

$$k_{22}^{(3)} = 3$$

$$q_1^{(3)} = -u'\left(\frac{2}{3}\right)$$

$$q_2^{(3)} = u'(1)$$

Doing the assembly:

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \cdot \begin{bmatrix} \cdot \\ u_2 \\ u_3 \\ \cdot \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + q_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} + \cancel{q_2^{(1)}} + \cancel{q_1^{(2)}} \\ f_2^{(2)} + f_1^{(3)} + \cancel{q_2^{(2)}} + \cancel{q_1^{(3)}} \\ f_2^{(3)} + q_2^{(3)} \end{bmatrix};$$

$$; \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - 3\sin\left(\frac{1}{3}\right) - u'(0) \\ 6\sin\left(\frac{1}{3}\right) - 3\sin\left(\frac{2}{3}\right) \\ 6\sin\left(\frac{2}{3}\right) - 3\sin\left(\frac{1}{3}\right) - 3\sin(1) \\ -\cos(1) + 3\sin(1) - 3\sin\left(\frac{2}{3}\right) + u'(1) \end{bmatrix}$$

Solving with maple we obtain:

$$\begin{bmatrix} u_2 = 1{,}0467 & u'(0) = 3'1585 \\ u_3 = 2{,}0574 & u'(1) = 2'1585 \end{bmatrix} ; \begin{bmatrix} u_{2\,exact} = 1{,}0467 & \text{No error} \\ u_{3\,exact} = 2{,}057 & \varepsilon = u - u_{exact} = 0 \end{bmatrix}$$